

# MODULAR REPRESENTATIONS OF HECKE ALGEBRAS

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**ABSTRACT.** These notes are based on a course given at the EPFL in May 2005. It is concerned with the representation theory of Hecke algebras in the non-semisimple case. We explain the role that these algebras play in the modular representation theory of finite groups of Lie type and survey the recent results which complete the classification of the simple modules. These results rely on the theory of Kazhdan–Lusztig cells in finite Weyl groups (with respect to possibly unequal parameters) and the theory of canonical bases for representations of quantum groups.

## 1. INTRODUCTION

The theory and the results that we are going to talk about can be seen as a contribution to the general project of determining the irreducible representations of all finite simple groups. Recall that such a group is either cyclic of prime order, or an alternating group of degree  $\geq 5$ , or a simple group of Lie type, or one of 26 sporadic simple groups. Here, we will concentrate on the finite groups of Lie type; any such group is naturally defined over a finite field  $\mathbb{F}_q$ , where  $q$  is a prime power. We shall consider representations over a field whose characteristic is a prime number not dividing  $q$ , the “non-defining characteristic case”. Iwahori–Hecke algebras associated with finite Coxeter groups naturally arise in this context, as endomorphism algebras of certain induced representations. This will be explained in Sections 2 and 3, where we give an introduction to Harish–Chandra series in the “modular case”. For further details and references, we refer to the survey [31].

Now the endomorphism algebras arising in this context can be defined abstractly, in terms of generators and relations. We will study the representations of Iwahori–Hecke algebras in this abstract setting, without reference to the realization as an endomorphism algebra of some induced representation; see Section 4. Our main aim is to explain a natural parametrization of the irreducible representations in terms of so-called “canonical basic sets” and the unitriangularity of the decomposition matrix, following work of Rouquier and the author [29], [48], [32]. This heavily relies on the theory of Kazhdan–Lusztig cells; see Sections 5 and 6.

Originally, we only considered “canonical basic sets” for Iwahori–Hecke algebras with equal parameters. Here, we present this theory in the general framework of possibly unequal parameters, following Lusztig [73]. Some

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properties of the Kazhdan–Lusztig basis are only conjectural in the general case, but the recent work of Bonnafé, Iancu and the author [6], [5], [43], [37] shows that these conjectural properties hold (at least) in the so-called “asymptotic case” in type  $B_n$ . For example, the characterization of the canonical basic set in this case in Example 6.9 is a new result.

Once the existence of a natural parametrization of the simple modules is established, it is another question to determine those “canonical basic sets” explicitly. In type  $B_n$  (in many ways the hardest case) this was achieved by Jacon [55], [57], [59], using the theory of canonical bases of quantum groups and building on earlier work of Ariki [1], [3] (the proof of the LLT conjecture), Ariki–Mathas [4], Foda et al. [25], and Uglov [81]. These results guarantee the existence of a “canonical basic set” even for those choices of unequal parameters where Lusztig’s conjectural properties are not (yet) known to hold. Thus, quite remarkably, irreducible representations of Iwahori–Hecke algebras of classical type are naturally labelled by the crystal bases of certain highest weight modules for the quantized enveloping algebra  $U_v(\hat{\mathfrak{sl}}_l)$ . All this will be discussed in Sections 7 and 8.

## 2. HARISH-CHANDRA SERIES AND HECKE ALGEBRAS

Let  $G$  be a finite group with a split BN-pair of characteristic  $p$  which satisfies Chevalley’s commutator relations; see [12, §2.5–2.6] or [35, §1.6]. We don’t repeat all the axioms here, but just recall some basic results. We have  $B = UH$  where  $U = O_p(B)$  is the largest normal  $p$ -subgroup of  $B$  and  $H$  is an abelian  $p'$ -subgroup such that  $H = B \cap N$ . Let  $W = N/H$  be the Weyl group of  $G$ . For any  $w \in W$  we denote by  $\dot{w}$  a representative in  $N$ . The group  $W$  is a Coxeter group with respect to the set of generators

$$S = \{w \in W \mid w \neq 1 \text{ and } B \cup B\dot{w}B \text{ is a subgroup of } G\}.$$

For any subset  $I \subseteq S$ , we have a corresponding parabolic subgroup  $W_I = \langle I \rangle \subseteq W$ . Moreover,  $I$  also defines a parabolic subgroup  $P_I = BN_I B \subseteq G$ , where  $N_I = \{H\dot{w} \mid w \in W_I\}$ . We have a Levi decomposition  $P_I = U_I L_I$  where  $U_I = O_p(P_I)$  is the largest normal  $p$ -subgroup of  $P_I$  and  $L_I$  is a complementary subgroup which is uniquely determined by the condition that  $H \subseteq L_I$ . The group  $L_I$  is called a standard Levi subgroup of  $G$ ; it is itself a finite group with a split BN-pair of characteristic  $p$  corresponding to the subgroups  $B_I = B \cap L_I$  and  $N_I = \{H\dot{w} \mid w \in W_I\}$ .

**Example 2.1.** Finite groups with a split BN-pair typically arise as the fixed point sets of connected reductive algebraic groups under a Frobenius map. (A formal definition of such maps is contained in [35, §4.1].) Here is the standard example. Let  $p$  be a prime and  $\overline{\mathbb{F}}_p$  be an algebraic closure of the finite field with  $p$  elements. Let  $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$  be the general linear group of  $n \times n$ -matrices over  $\overline{\mathbb{F}}_p$ . Let  $\mathbf{B} \subseteq \mathbf{G}$  be the subgroup consisting of all upper triangular matrices, and  $\mathbf{N} \subseteq \mathbf{G}$  be the subgroup consisting of all monomial matrices. Then it is well-known that the groups  $\mathbf{B}$  and  $\mathbf{N}$  form a BN-pair

in  $\mathbf{G}$ , with Weyl group  $\mathbf{W} \cong \mathfrak{S}_n$ , the symmetric group on  $n$  letters; see [35, §1.6] for more details. We have a semidirect product decomposition  $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}_0$  where  $\mathbf{U}$  is the normal subgroup consisting of all upper triangular matrices with 1 on the diagonal, and  $\mathbf{T}_0$  is the subgroup consisting of all invertible diagonal matrices; then  $\mathbf{N} = \mathbf{N}_{\mathbf{G}}(\mathbf{T}_0)$ , the normalizer of  $\mathbf{T}_0$  in  $\mathbf{G}$ , and  $\mathbf{W} = \mathbf{N}/\mathbf{T}_0$ .

(a) Let  $q = p^f$  for some  $f \geq 1$ . Then we have a unique subfield  $\mathbb{F}_q \subseteq \overline{\mathbb{F}}_p$  with  $q$  elements. We have the “standard” Frobenius map

$$F_q: \mathbf{G} \rightarrow \mathbf{G}, \quad (a_{ij}) \mapsto (a_{ij}^q).$$

The group of fixed points is  $\mathbf{G}^{F_q} = \mathrm{GL}_n(\mathbb{F}_q)$ , the general linear group over  $\mathbb{F}_q$ . The groups  $\mathbf{B}$  and  $\mathbf{N}$  are  $F_q$ -stable. By taking fixed points under  $F_q$ , we obtain that  $\mathbf{B}^{F_q}$  and  $\mathbf{N}^{F_q}$  form a split BN-pair of characteristic  $p$  in the finite group  $\mathrm{GL}_n(\mathbb{F}_q)$ , with Weyl group  $W \cong \mathfrak{S}_n$ ; see [35, §4.2].

(b) Now consider the permutation matrix

$$Q_n := \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in M_n(k)$$

and define an automorphism of algebraic groups  $\gamma: \mathbf{G} \rightarrow \mathbf{G}$  by

$$\gamma(A) := Q_n^{-1} \cdot (A^{\mathrm{tr}})^{-1} \cdot Q_n \quad \text{where } A \in \mathrm{GL}_n(\overline{\mathbb{F}}_p).$$

Then  $\gamma$  commutes with  $F_q$  and  $\gamma^2$  is the identity. Hence, the map  $F := F_q \circ \gamma$  also is a Frobenius map on  $\mathbf{G}$ . Since  $F^2$  is the standard Frobenius map with respect to  $\mathbb{F}_{q^2}$ , we have  $\mathbf{G}^F \subseteq \mathrm{GL}_n(\mathbb{F}_{q^2})$ . Now the restriction of  $F_q$  to  $\mathrm{GL}_n(\mathbb{F}_{q^2})$  is an automorphism of order 2, which we denote by  $A \mapsto \bar{A}$ . Then we obtain

$$\mathbf{G}^F = \mathrm{GU}_n(\mathbb{F}_q) := \{A \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid \bar{A}^{\mathrm{tr}} \cdot Q_n \cdot A = Q_n\},$$

the general unitary group with respect to the hermitian form defined by  $Q_n$ . The groups  $\mathbf{B}$  and  $\mathbf{N}$  are  $F$ -stable; furthermore,  $F$  induces an automorphism of  $\mathbf{W} \cong \mathfrak{S}_n$  which we denote by the same symbol; that automorphism is given by conjugation with the permutation  $w_0 \in \mathfrak{S}_n$  whose matrix is  $Q_n$ . Then  $\mathbf{B}^F$  and  $\mathbf{N}^F$  form a split BN-pair of characteristic  $p$  in  $\mathrm{GU}_n(\mathbb{F}_q)$ , with Weyl group  $W \cong \mathbf{W}^F$ ; see [35, 4.2.6]. We have that  $W$  is a Coxeter group of type  $B_{n/2}$  (if  $n$  is even) or of type  $B_{(n-1)/2}$  (if  $n$  is odd).

Now let  $k$  be an algebraically closed field such that  $p$  is invertible in  $k$ . We denote by  $kG\text{-mod}$  the category of finite-dimensional (left)  $kG$ -modules. Let  $\mathrm{Irr}_k(G)$  be the set of simple  $kG$ -modules (up to isomorphism). Now let  $I \subseteq S$ . As we have noted above, the Levi subgroup  $L_I$  is like  $G$ , that is, a group with a split BN-pair of characteristic  $p$ . We define functors

$$R_I^S: kL_I\text{-mod} \rightarrow kG\text{-mod} \quad \text{and} \quad {}^*R_I^S: kG\text{-mod} \rightarrow kL_I\text{-mod}$$

as follows. Let  $X \in kL_I\text{-mod}$ . Then we can regard  $X$  as a  $kP_I$ -module via the canonical map  $P_I \rightarrow L_I$  with kernel  $U_I$ ; denote that  $kP_I$ -module by  $\tilde{X}$ . Then we set  $R_I^S(X) = \text{Ind}_{P_I}^G(\tilde{X})$  (where  $\text{Ind}$  denotes the usual induction of modules from subgroups).

Conversely, let  $Y \in kG\text{-mod}$ . Then, since  $U_I$  is normalized by  $L_I$ , the fixed point set  $\text{Fix}_{U_I}(Y)$  is naturally a  $kL_I$ -module, which we denote  $*R_I^S(Y)$ . Since  $p$  is invertible in  $k$ , we have the following alternative description. Let

$$e_I = \frac{1}{|U_I|} \sum_{u \in U_I} u \in kG;$$

i.e.,  $e_I$  is the idempotent corresponding to the trivial  $kU_I$ -module. Then the map  $\text{Fix}_{U_I}(Y) \rightarrow e_I Y$ ,  $y \mapsto e_I y$ , is an isomorphism of  $kL_I$ -modules.

The functors  $R_I^S$  and  $*R_I^S$  have functorial properties similar to the usual induction and restriction: transitivity, adjointness with respect to  $\text{Hom}$  and a Mackey formula. See the survey in [31] for further details and references.

**Definition 2.2.** Let  $Y \in kG\text{-mod}$ . We say that  $Y$  is *cuspidal* if  $*R_I^S(Y) = \{0\}$  for every proper subset  $I \subset S$ . Thus,  $Y$  is cuspidal if and only if  $\text{Fix}_{U_I}(Y) = \{0\}$  for every proper  $I \subset S$ . We set

$$\text{Irr}_k^\circ(G) := \{Y \in \text{Irr}_k(G) \mid Y \text{ is cuspidal}\}.$$

Similar notations are used for Levi subgroups of  $G$ .

Let  $\mathfrak{C}_G$  be the set of all pairs  $(I, X)$  where  $I \subseteq S$  and  $X \in \text{Irr}_k^\circ(L_I)$ . Given two such pairs  $(I, X)$  and  $(J, Y)$ , we write  $(I, X) \approx (J, Y)$  if there exists some  $w \in W$  such that  $w^{-1}Iw = J$  and  ${}^wX \cong_{kL_J} Y$ . (Here,  ${}^wX$  is the  $kL_J$ -module with the same underlying vector space  $X$  but where the action is defined by composition with the group isomorphism  $L_I \rightarrow L_J$  given by conjugation with  $w$ .) This defines an equivalence relation on  $\mathfrak{C}_G$ .

Now let us fix  $Y \in \text{Irr}_k(G)$ . Then, by the transitivity of Harish–Chandra induction and restriction, there exists a pair  $(I, X) \in \mathfrak{C}_G$  such that the following two conditions are satisfied:

- (HC1)  $*R_I^S(Y) \neq \{0\}$  and  $I \subseteq S$  is minimal with property;
- (HC2)  $X$  is a composition factor of  $*R_I^S(Y)$ .

It is known that, if  $(I', X') \in \mathfrak{C}_G$  is another pair satisfying (HC1) and (HC2), then  $(I, X) \approx (I', X')$ ; see Hiss [52] for the case where  $k$  has positive characteristic. This leads to the following definition. Let  $(I, X) \in \mathfrak{C}_G$ . Then  $\text{Irr}_k(G, (I, X))$  is defined to be the set of all  $Y \in \text{Irr}_k(G)$  such that (HC1) and (HC2) hold. The set  $\text{Irr}_k(G, (I, X))$  is called the *Harish–Chandra series* defined by  $(I, X)$ . We have

$$\text{Irr}_k(G) = \coprod_{(I, X) \in \mathfrak{C}_G / \approx} \text{Irr}_k(G, (I, X)).$$

Thus, up to this point, everything formally works as in the classical theory of Harish–Chandra series in characteristic zero; see, for example, Carter [12, Chap. 9] or Digne–Michel [15, Chap. 6]. In this case, by the adjointness

between Harish–Chandra induction and restriction, the series  $\text{Irr}_k(G, (I, X))$  can simply be characterized as the set of all simple  $kG$ -modules which occur as composition factors in  $R_I^S(X)$ . In the general case where  $k$  is no longer assumed to have characteristic zero, we have the following result:

**Theorem 2.3** (Hiss [52]). *Let  $Y \in \text{Irr}_k(G)$  and  $(I, X) \in \mathfrak{C}_G$ . Then the following three conditions are equivalent:*

- (a)  $Y \in \text{Irr}_k(G, (I, X))$ .
- (b)  $Y$  is isomorphic to a submodule of  $R_I^S(X)$ .
- (c)  $Y$  is isomorphic to a quotient of  $R_I^S(X)$ .

The above formulation of Hiss’ result relies on the following fundamental property of Harish–Chandra induction:

$$R_I^S(X) \cong_{kG} R_J^S(Y) \quad \text{if } (I, X) \approx (J, Y) \text{ in } \mathfrak{C}_G.$$

This was proved independently by Dipper–Du [17] and Howlett–Lehrer [54].

**Example 2.4.** Let  $G = \text{GL}_2(\mathbb{F}_q)$  where  $q$  is a power of the prime  $p$ , with the BN-pair specified in Example 2.1. Consider the pair  $(\emptyset, k_H) \in \mathfrak{C}_G$ , where  $k_H$  denotes the trivial  $kH$ -module. Then  $M := R_\emptyset^S(k_H)$  is the permutation module on the cosets of  $B$ ; in particular, we have

$$\dim M = [G : B] = q + 1.$$

There are two essentially different cases:

- The characteristic of  $k$  does not divide  $[G : B]$ . Then

$$M \cong_{kG} k_G \oplus Y \quad \text{where } Y \in \text{Irr}_k(G), \dim Y = q;$$

in fact,  $Y$  is the Steinberg module. Thus we have

$$\text{Irr}_k(G, (\emptyset, k_H)) = \{k_G, Y\}.$$

- The characteristic of  $k$  divides  $[G : B]$ . Then  $M$  is indecomposable with submodules  $\{0\} \subset V \subset V' \subset M$  such that

$$V \cong_{kG} M/V' \cong_{kG} k_G \quad \text{and} \quad V'/V \in \text{Irr}_k^\circ(G).$$

Hence, in this case, we have  $\text{Irr}_k(G, (\emptyset, k_H)) = \{k_G\}$ .

Now let us return to the general situation and fix a pair  $(I, X) \in \mathfrak{C}_G$ . In order to obtain more information about the corresponding Harish–Chandra series  $\text{Irr}_k(G, (I, X))$ , we consider

$$\mathcal{H} := \text{End}_{kG}(R_I^S(X))^\circ,$$

the opposite algebra of the algebra of  $kG$ -endomorphisms of  $R_I^S(X)$ . This is a finite-dimensional algebra over  $k$ , called a *Hecke algebra*. We denote by  $\mathcal{H}\text{-mod}$  the category of finite-dimensional (left)  $\mathcal{H}$ -modules and by  $\text{Irr}(\mathcal{H})$  the set of simple  $\mathcal{H}$ -modules, up to isomorphism. We have a functor

$$\mathfrak{F}: kG\text{-mod} \rightarrow \mathcal{H}\text{-mod}, \quad V \mapsto \text{Hom}_{kG}(R_I^S(X), V),$$

where the action of  $\varphi \in \mathcal{H}$  on  $\mathfrak{F}(V)$  is given by  $\varphi.f = f \circ \varphi$  for  $f \in \mathfrak{F}(V)$ , and where  $\mathfrak{F}$  sends a homomorphism of  $kG$ -modules  $\rho: V \rightarrow V'$  to the homomorphism of  $\mathcal{H}$ -modules  $\mathfrak{F}(\rho) = \rho_*: \mathfrak{F}(V) \rightarrow \mathfrak{F}(V')$ ,  $f \mapsto \rho \circ f$ .

By general principles (“Fitting correspondence”),  $\mathfrak{F}$  induces a bijection between the isomorphism classes of indecomposable direct summands of  $R_I^S(X)$  and the isomorphism classes of projective indecomposable  $\mathcal{H}$ -modules; note that the latter are in natural bijection with  $\text{Irr}(\mathcal{H})$ . Hence, if  $k$  has characteristic zero, then  $\mathfrak{F}$  certainly induces a bijection between  $\text{Irr}_k(G, (I, X))$  and  $\text{Irr}(\mathcal{H})$ . This statement remains valid in the general case, but the proof is much harder: one has to establish some properties of the indecomposable direct summands of  $R_I^S(X)$ . The precise result is as follows.

**Theorem 2.5** (Geck–Hiss–Malle [42], Geck–Hiss [40]). *In the above setting,  $\mathcal{H}$  is a symmetric algebra and we have a bijection*

$$\text{Irr}_k(G, (I, X)) \xrightarrow{\sim} \text{Irr}(\mathcal{H}), \quad Y \mapsto \mathfrak{F}(Y).$$

*Every indecomposable direct summand of  $R_I^S(X)$  has a unique simple submodule and a unique simple quotient, and these are isomorphic to each other.*

The fact that  $\mathcal{H}$  is symmetric means that there exists a trace function  $\tau: \mathcal{H} \rightarrow k$  such that the associated bilinear form

$$\mathcal{H} \times \mathcal{H} \rightarrow k, \quad (h, h') \mapsto \tau(hh'),$$

is non-degenerate. The existence of  $\tau$  is proved in [40], using the construction of a suitable basis of  $\mathcal{H}$  in [42]; see the remarks further below. Thus,  $R_I^S(X)$  is a module whose endomorphism algebra is symmetric and which satisfies the two equivalent conditions (b) and (c) in Theorem 2.3. Then the statements in Theorem 2.5 follow from a general argument combining ideas of Green, Cabanes and Linckelmann; see [31, §2 and §3] for further details. It is interesting to note that these arguments first appeared in the representation theory of finite groups with a split BN-pair of characteristic  $p$ , where the base field of the representations also has characteristic  $p$ .

Finally, we describe the structure of  $\mathcal{H}$  in some more detail. For this purpose, we recall the general definition of an Iwahori–Hecke algebra associated with a Coxeter group; see [47] for the general theory. Let  $W_1$  be a finite Coxeter group. Thus,  $W_1$  has a presentation with generators  $S_1 \subseteq W_1$  and defining relations of the form:

- $s^2 = 1$  for all  $s \in S_1$ ;
- $(st)^{m(s,t)} = 1$  for  $s \neq t$  in  $S_1$ , where  $m(s, t)$  denotes the order of  $st$ .

Let  $l: W_1 \rightarrow \mathbb{N}$  be the corresponding length function, where  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Let  $R$  be any commutative ring with 1, and let  $R^\times$  be the group of multiplicative units. We say that a function

$$\pi: W_1 \rightarrow R^\times$$

is a parameter function if  $\pi(w w') = \pi(w) \cdot \pi(w')$  whenever we have  $l(w w') = l(w) + l(w')$  for  $w, w' \in W_1$ . Such a function is uniquely determined by

the values  $\pi(s)$ ,  $s \in S_1$ , subject only to the condition that  $\pi(s) = \pi(t)$  if  $s, t \in S_1$  are conjugate in  $W_1$ . Let  $H_1 = H_R(W_1, \pi)$  be the corresponding Iwahori–Hecke algebra over  $R$  with parameters  $\{\pi(s) \mid s \in S_1\}$ . The algebra  $H_1$  is free over  $R$  with basis  $\{T_w \mid w \in W_1\}$ , and the multiplication is given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ \pi(s)T_{sw} + (\pi(s) - 1)T_w & \text{if } l(sw) < l(w), \end{cases}$$

where  $s \in S_1$  and  $w \in W_1$ . We define a linear map  $\tau: H_1 \rightarrow k$  by

$$\tau(T_1) = 1 \quad \text{and} \quad \tau(T_w) = 0 \quad \text{for } w \neq 1.$$

Then one can show that  $\tau$  is a trace function and we have

$$\tau(T_w T_{w'}) = \begin{cases} \pi(w) & \text{if } w' = w^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that  $H$  is a symmetric algebra.

Now let us return to the pair  $(I, X) \in \mathfrak{C}_G$  and the corresponding endomorphism algebra  $\mathcal{H}$ . Let  $\mathcal{N}(I, X)$  be the stabilizer of  $X$  in  $\mathcal{N}(I) := (N_G(L_I) \cap N)L_I$ . Then we have

$$\dim \mathcal{H} = |\mathcal{W}(I, X)| \quad \text{where} \quad \mathcal{W}(I, X) := \mathcal{N}(I, X)/L.$$

In fact, one can explicitly construct a basis  $\{B_w \mid w \in \mathcal{W}(I, X)\}$  of  $\mathcal{H}$ , as in [42]; that construction generalizes the one by Howlett–Lehrer [53] (which is concerned with the case where the ground field has characteristic zero). Now, there is a semidirect product decomposition  $\mathcal{W}(I, X) = W_1 \rtimes \Omega$  where  $W_1$  is a finite Coxeter group (with generating set  $S_1$ ) and  $\Omega$  is a finite group such that  $\omega S_1 \omega^{-1} = S_1$  for all  $\omega \in \Omega$ . Let  $\mathcal{H}_1 := \langle B_w \mid w \in W_1 \rangle \subseteq \mathcal{H}$ . The multiplicative properties of the basis  $\{B_w\}$  show that  $\mathcal{H}_1$  is a subalgebra of  $\mathcal{H}$  and that we have a direct sum decomposition

$$\mathcal{H} = \bigoplus_{\omega \in \Omega} \mathcal{H}_\omega \quad \text{where} \quad \mathcal{H}_\omega := H_1 \cdot B_\omega = B_\omega \cdot \mathcal{H}_1;$$

furthermore, each  $B_\omega$  ( $\omega \in \Omega$ ) is invertible and we have  $\mathcal{H}_{\omega\omega'} = \mathcal{H}_\omega \cdot B_{\omega'}$  for all  $\omega, \omega' \in \Omega$ . Hence, as remarked in [40, Prop. 2.4], the family of subspaces  $\{\mathcal{H}_\omega \mid \omega \in \Omega\}$  is an  $\Omega$ -graded Clifford system in  $\mathcal{H}$ , in the sense of [13, Def. 11.12].

**Theorem 2.6** (Howlett–Lehrer [53] ( $\text{char}(k)=0$ ) and Geck–Hiss–Malle [42]). *In the above setting, assume that  $X$  can be extended to  $\mathcal{N}(I, X)$ . Then we have  $\mathcal{H}_1 \cong H_k(W_1, \pi)$  for a suitable parameter function  $\pi: W_1 \rightarrow k^\times$ .*

In many cases, we have  $\Omega = \{1\}$  and the hypothesis of the above result can be seen to hold. Hence, in these cases, the algebra  $\mathcal{H}_1$  is an Iwahori–Hecke algebra associated with a finite Coxeter group. We will see some examples in the following section.

*Remark 2.7.* Assume that the hypotheses of Theorem 2.6 are satisfied, where  $k$  is an algebraic closure of the finite field with  $\ell$  elements. Thus, we have  $\mathcal{H}_1 \cong H_k(W_1, \pi)$  where  $\pi: W_1 \rightarrow k^\times$  is a parameter function. Since  $W_1$  is finite, there exists a finite subfield  $k_0 \subseteq k$  such that  $\pi(W) \subseteq k_0$ . Hence, since  $k_0^\times$  is cyclic, there exist some  $\xi \in k^\times$  and  $d \geq 1$  such that

$$\xi^d = 1 \quad \text{and} \quad \pi(s) = \xi^{L(s)} \quad \text{for all } s \in S_1,$$

where  $L: W_1 \rightarrow \mathbb{N}$  is a *weight function* in the sense of Lusztig [73], that is, we have  $L(ww') = L(w) + L(w')$  whenever we have  $l(ww') = l(w) + l(w')$  for  $w, w' \in W_1$ .

*Remark 2.8.* Dipper and James extensively studied the case where  $G = \mathrm{GL}_n(\mathbb{F}_q)$  and  $k$  has positive characteristic; see [16], [60] and the references there. They obtained a complete classification of all cuspidal simple modules and a parametrization of the simple modules of the corresponding Hecke algebras in this case. An outstanding role in this context is played by the introduction of the  $q$ -Schur algebra [20]. A special feature of  $\mathrm{GL}_n(\mathbb{F}_q)$  is the fact that all cuspidal simple modules in positive characteristic have a “reduction stable” lift to characteristic zero. The fact that this is no longer true for groups of other types is the source of substantial, new complications.

### 3. UNIPOTENT BLOCKS

We now wish to restrict our attention to the “unipotent” modules of  $G$ , in the sense of Deligne–Lusztig [14]. For this purpose, let us assume from now on that  $G$  arises as the fixed point set of a connected reductive algebraic group  $\mathbf{G}$  over  $\overline{\mathbb{F}}_p$  under a Frobenius map  $F: \mathbf{G} \rightarrow \mathbf{G}$ , with respect to some  $\mathbb{F}_q$ -rational structure on  $G$  where  $q$  is a power of  $p$ . (See Example 2.1.) Thus, we have

$$G = \mathbf{G}(\mathbb{F}_q) = \mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}.$$

Let  $\mathbf{B} \subseteq \mathbf{G}$  be an  $F$ -stable Borel subgroup and  $\mathbf{T}_0 \subseteq \mathbf{G}$  be an  $F$ -stable maximal torus which is contained in  $\mathbf{B}$ . Let  $\mathbf{W}$  be the Weyl group of  $\mathbf{G}$  with respect to  $\mathbf{T}_0$ ; the Frobenius map  $F$  induces an automorphism of  $\mathbf{W}$  which we denote by the same symbol. Then  $G$  is a finite group with a split  $BN$ -pair of characteristic  $p$  and Weyl group  $W$  where

$$B := \mathbf{B}^F, \quad N := N_{\mathbf{G}}(\mathbf{T}_0)^F, \quad W \cong \mathbf{W}^F;$$

see [12, §2.9].

We shall need some results about the modules of  $G$  over the field  $\mathbb{C}$ . Given  $V \in \mathbb{C}G\text{-mod}$ , the corresponding character is the function  $\chi_V: G \rightarrow \mathbb{C}$  which sends  $g \in G$  to the trace of  $g$  acting on  $V$ . It is well-known that  $V \cong_{\mathbb{C}G} V'$  (where  $V, V' \in \mathbb{C}G\text{-mod}$ ) if and only if  $\chi_V = \chi_{V'}$ . We set

$$G^\wedge = \{\chi: G \rightarrow \mathbb{C} \mid \chi = \chi_V \text{ for some } V \in \mathrm{Irr}_{\mathbb{C}}(G)\}.$$

A function  $f: G \rightarrow \mathbb{C}$  will be called a *virtual character* if  $f$  is an integral linear combination of  $G^\wedge$ . In this case, we denote by  $\langle f: \chi \rangle$  the coefficient



of  $\chi$  in the expansion of  $f$  in terms of  $G^\wedge$ , that is, we have

$$f = \sum_{\chi \in G^\wedge} \langle f : \chi \rangle \chi.$$

Recall that the conjugates of  $\mathbf{T}_0$  in  $\mathbf{G}$  are called the *maximal tori* of  $\mathbf{G}$ . Let  $g \in \mathbf{G}$  and consider  $\mathbf{T} := g\mathbf{T}_0g^{-1}$ . This group will be  $F$ -stable if and only if  $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T}_0)$ . In this case,  $F$  restricts to a Frobenius map on  $\mathbf{T}$  and, by taking fixed points, we obtain a finite group  $T := \mathbf{T}^F \subseteq G$ . Note that  $T$  is abelian of order prime to  $p$ . Consequently, the set  $T^\wedge$  also is an abelian group, for the pointwise product of characters.

Let  $\mathcal{S}$  be the set of all pairs  $(\mathbf{T}, \theta)$  where  $\mathbf{T} \subseteq \mathbf{G}$  is an  $F$ -stable maximal torus and  $\theta \in T^\wedge$ . Deligne and Lusztig [14] have associated with each pair  $(\mathbf{T}, \theta) \in \mathcal{S}$  a virtual character  $R_{T, \theta}$  of  $G$ . The construction uses deep results from algebraic geometry; see Lusztig [68] and Carter [12] for more details. Here are some properties:

- if  $\mathbf{T} = \mathbf{T}_0$ , then  $R_{T, \theta}$  is the character of  $R_{\mathcal{O}}^S(\theta)$ ;
- $R_{T, \theta}(1) = \eta_T [G : T]_{p'}$ , where  $\eta_T = \pm 1$ ;
- given any  $\chi \in G^\wedge$ , we have  $\langle R_{T, \theta} : \chi \rangle \neq 0$  for some  $(\mathbf{T}, \theta) \in \mathcal{S}$ .

Now let  $\mathcal{S}_1$  be the set of all pairs  $(\mathbf{T}, \theta) \in \mathcal{S}$  such that  $\theta$  is the principal character  $1_T$  of  $T$ . Then the characters

$$\text{Uch}(G) := \{\chi \in G^\wedge \mid \langle R_{T, \theta} : \chi \rangle \neq 0 \text{ for some } (T, \theta) \in \mathcal{S}_1\}$$

are called the *unipotent characters* of  $G$ . Lusztig [69], [71] obtained a classification of the unipotent characters and formulae for the corresponding degrees; furthermore, he showed that the classification of all irreducible characters can be reduced to the case of unipotent characters, in terms of a “Jordan decomposition” of characters. The unipotent characters only depend on the Weyl group of  $\mathbf{G}$ ; more precisely:

**Theorem 3.1** (Lusztig [69]). *There exists a finite set  $\Lambda$  and polynomials  $D_\lambda \in \mathbb{Q}[X]$  ( $\lambda \in \Lambda$ ) such that there is a bijection*

$$\Lambda \xrightarrow{\sim} \text{Uch}(G), \quad \lambda \mapsto \chi_\lambda,$$

where  $\chi_\lambda(1) = D_\lambda(q)$  for all  $\lambda \in \Lambda$ . The set  $\Lambda$  and the polynomials  $\{D_\lambda\}$  only depend on the Coxeter group  $\mathbf{W}$  and the induced map  $F : \mathbf{W} \rightarrow \mathbf{W}$ .

Explicit tables of the polynomials  $D_\lambda$  can be found in the appendix of [69]. Thus, the unipotent characters play an essential role in the whole theory. An analogous result in the “modular situation” has been obtained by Bonnafé–Rouquier [7]. The definition of unipotent  $kG$ -modules, where  $k$  has positive characteristic  $\ell \neq p$ , is based on the following result.

**Theorem 3.2** (Broué–Michel [11]). *Let  $\ell$  be a prime number  $\neq p$  and set*

$$b_\ell(g) := \frac{1}{|G|} \frac{1}{|G|_p} \sum_{(\mathbf{T}, \theta)} \eta_T R_{T, \theta}(g^{-1}) \in \mathbb{C} \quad \text{for any } g \in G,$$

where the sum runs over all pairs  $(\mathbf{T}, \theta) \in \mathcal{S}$  such that  $\theta$  is of order a power of  $\ell$  in  $T^\wedge$ . Then we have  $b_\ell(g) \in \mathbb{Z}_{(\ell)}$  for any  $g \in G$ , where  $\mathbb{Z}_{(\ell)}$  denotes the localization of  $\mathbb{Z}$  in the prime ideal generated by  $\ell$ . The element

$$\beta_\ell := \sum_{g \in G} \bar{b}_\ell(g) g \in \mathbb{F}_\ell G$$

is a central idempotent, where the bar denotes the canonical map  $\mathbb{Z}_{(\ell)} \rightarrow \mathbb{F}_\ell$ .

Let us indicate how the formula for  $b_\ell(g)$  comes about. For any  $\chi \in G^\wedge$ , let  $e_\chi$  be the corresponding central primitive idempotent in  $\mathbb{C}G$ . We have

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(1) \chi(g^{-1}) g.$$

Now let  $\mathcal{S}_\ell$  be the set of all pairs  $(\mathbf{T}, \theta) \in \mathcal{S}$  such that  $\theta$  is of order a power of  $\ell$  in  $T^\wedge$ . We set

$$e_\ell(g) := \frac{1}{|G|} \sum_{\chi \in \mathcal{B}_\ell} \chi(1) \chi(g^{-1}) \quad \text{for any } g \in G,$$

where  $\mathcal{B}_\ell$  is the set of all  $\chi \in G^\wedge$  such that  $\langle R_{T,\theta} : \chi \rangle \neq 0$  for some  $(\mathbf{T}, \theta) \in \mathcal{S}_\ell$ . Hence,  $e_\ell := \sum_{g \in G} e_\ell(g) g$  is a central idempotent in  $\mathbb{C}G$ . We want to show that  $e_\ell(g) = b_\ell(g)$  for all  $g \in G$ . For this purpose, we consider the character  $\chi_{\text{reg}}$  of the regular representation of  $G$ . We have

$$\chi_{\text{reg}} = \sum_{\chi \in G^\wedge} \chi(1) \chi = \frac{1}{|G|_p} \sum_{(\mathbf{T}, \theta) \in \mathcal{S}} \eta_T R_{T,\theta},$$

where the first equality is well-known and the second equality holds by [12, Cor. 7.5.6]. Using the partition of  $G^\wedge$  into geometric conjugacy classes (see [12, §7.3]), we conclude that

$$\sum_{\chi \in \mathcal{B}_\ell} \chi(1) \chi = \frac{1}{|G|_p} \sum_{(\mathbf{T}, \theta) \in \mathcal{S}_\ell} \eta_T R_{T,\theta},$$

Thus, we have  $e_\ell(g) = b_\ell(g)$  for all  $g \in G$ , as claimed. In particular, we have shown that  $b_\ell := \sum_{g \in G} b_\ell(g) g$  is a central idempotent in  $\mathbb{C}G$ .

Finally, using the character formula for  $R_{T,\theta}$  (see [12, 7.2.8]) one easily shows that  $b_\ell(g) \in \mathbb{Q}$  for all  $g \in G$ . Now Broué–Michel prove that the coefficients actually lie in  $\mathbb{Z}_{(\ell)}$ . Hence, we can reduce  $b_\ell$  modulo  $\ell$  and obtain a central idempotent  $\beta_\ell \in \mathbb{F}_\ell G$ , as desired.  $\square$

Now let  $k$  be an algebraically closed field of characteristic  $\ell$ , where  $\ell$  is a prime  $\neq p$ . Then  $\beta_\ell \in \mathbb{F}_\ell G \subseteq kG$  is a central idempotent and, hence, we have a direct sum decomposition

$$kG = \beta_\ell kG \oplus (1 - \beta_\ell) kG$$

where both  $\beta_\ell kG$  and  $(1 - \beta_\ell) kG$  are two-sided ideals. Correspondingly, we have a decomposition of  $kG$ -mod into those modules on which  $\beta_\ell$  acts as

the identity on the one hand, and those modules on which  $\beta_\ell$  acts as zero on the other hand. We set

$$\mathrm{Uch}_\ell(G) := \{Y \in \mathrm{Irr}_k(G) \mid \beta_\ell.Y = Y\}$$

and

$$\mathrm{Uch}_\ell^\circ(G) := \mathrm{Uch}_\ell(G) \cap \mathrm{Irr}_k^\circ(G).$$

Note that, if we formally set  $\ell = 1$  and take  $k = \mathbb{C}$ , then  $\mathrm{Uch}_1(G)$  indeed is the set of simple  $\mathbb{C}G$ -modules whose character is unipotent. Thus, the above definitions generalize the definition of unipotent characters to  $kG$ -modules where the characteristic of  $k$  is a prime  $\ell \neq p$ . Now we can state:

**Corollary 3.3** (Hiss [52]). *Let  $k$  be an algebraically closed field of characteristic  $\ell$ , where  $\ell$  is a prime  $\neq p$ . Then we have*

$$\mathrm{Uch}_\ell(G) = \coprod_{(I,X)} \mathrm{Irr}_k(G, (I, X)),$$

where the union runs over all pairs  $(I, X) \in \mathfrak{C}_G$  such that  $X \in \mathrm{Uch}_\ell^\circ(L_I)$ .

This is based on a compatibility of Harish-Chandra induction with the operator  $R_{T,\theta}$ ; see Lusztig [67, Cor. 6].

**Theorem 3.4** (Geck–Hiss [39]). *Assume that the center of  $\mathbf{G}$  is connected and that  $\ell$  is good for  $\mathbf{G}$ . Then we have  $|\mathrm{Uch}_\ell(G)| = |\mathrm{Uch}(G)|$ .*

We say that the prime  $\ell$  is *good* for  $\mathbf{G}$  if it is good for each simple factor involved in  $\mathbf{G}$ ; the conditions for the various simple types are as follows.

$$\begin{aligned} A_n &: \text{no condition,} \\ B_n, C_n, D_n &: \ell \neq 2, \\ G_2, F_4, E_6, E_7 &: \ell \neq 2, 3, \\ E_8 &: \ell \neq 2, 3, 5. \end{aligned}$$

If  $\ell$  is not good for  $G$ , we have  $|\mathrm{Uch}_\ell(G)| \neq |\mathrm{Uch}(G)|$ . For further information on the cardinalities of the sets  $|\mathrm{Uch}_\ell(G)|$  in this case, see [40, 6.6] and [18, §4.1].

**Conjecture 3.5** (James for type  $\mathrm{GL}_n$ ). *Recall the finite set  $\mathbf{\Lambda}$  and the polynomials  $D_\lambda$  in Theorem 3.1. Now let  $e \geq 1$ . Then there exist polynomials  $D_\lambda^{(e)} \in \mathbb{Q}[X]$  ( $\lambda \in \mathbf{\Lambda}$ ) such that the following holds: for any prime  $\ell \neq p$  such that  $\ell$  does not divide  $|\mathbf{W}|$  and  $e$  is the multiplicative order of  $q$  modulo  $\ell$ , we have a bijection*

$$\mathbf{\Lambda} \xrightarrow{\sim} \mathrm{Uch}_\ell(G), \quad \lambda \mapsto X^\lambda,$$

such that  $\dim X^\lambda = D_\lambda^{(e)}(q)$  for any  $\lambda \in \mathbf{\Lambda}$ . The polynomials  $D_\lambda$  and  $D_\lambda^{(e)}$  have the same degree and the same leading coefficient.

The above conjecture is known to hold in some examples of small rank by explicit verification, most notably for  $G = \mathrm{GL}_n(\mathbb{F}_q)$  where  $n \leq 10$ ; see James [61]. For the example  $G = \mathrm{GU}_3(\mathbb{F}_q)$ ; see Okuyami–Waki [80].

Let us now turn to the discussion of some examples of cuspidal unipotent modules and the corresponding Hecke algebras.

**Example 3.6.** Assume that  $G = \mathbf{G}^F$  and that the induced map  $F: \mathbf{W} \rightarrow \mathbf{W}$  is the identity. The pair  $(\emptyset, k_H)$  is cuspidal and  $k_H$  is unipotent. Furthermore,  $R_{\emptyset}^S(k_H)$  is nothing but the permutation module of  $G$  on the cosets of  $B$ . This is the case originally considered by Iwahori; see [47, §8.4] and the references there. We have

$$\mathcal{H} \cong H_k(W, \bar{q})$$

where the notation indicates that the parameter function  $\pi: W \rightarrow k$  is given by  $\pi(s) = \bar{q} := q \cdot 1_k \in k$  for all  $s \in S$ .

**Example 3.7.** Let  $G = \mathrm{GL}_n(\mathbb{F}_q)$ . Then we have

$$|\mathrm{Uch}_{\ell}(G)| = |\mathrm{Uch}(G)| = \text{number of partitions of } n.$$

Now let  $e$  be the multiplicative order of  $q$  modulo  $\ell$ . Then we have

$$(*) \quad |\mathrm{Uch}_{\ell}^{\circ}(\mathrm{GL}_n(\mathbb{F}_q))| = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = e\ell^i \text{ for some } i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

There are several proofs for this result: see James [60], Dipper [16], Geck–Hiss–Malle [41, §7] or [42, §2]. The unique  $X^{\circ} \in \mathrm{Uch}_{\ell}^{\circ}(G)$  (if it exists) can be lifted to a cuspidal module in characteristic zero (which is not unipotent); we have

$$\dim X^{\circ} = (q^{n-1} - 1)(q^{n-2} - 1) \cdots (q^2 - 1)(q - 1).$$

Let us now consider the Hecke algebras arising in  $G$ . Let  $\lambda$  be a partition of  $n$ , with non-zero parts  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ . Then we have a corresponding Levi subgroup  $L_I \subseteq G$  such that

$$L_I \cong \mathrm{GL}_{\lambda_1}(\mathbb{F}_q) \times \cdots \times \mathrm{GL}_{\lambda_r}(\mathbb{F}_q);$$

furthermore, any Levi subgroup of  $G$  arises in this way (up to  $\approx$ ). Now, a simple module  $X \in \mathrm{Irr}_k(L_I)$  is isomorphic to a tensor product of simple modules for the various factors in the above direct product, and  $X$  is cuspidal unipotent if and only if every factor has this property. Let us assume that  $X \in \mathrm{Uch}_{\ell}^{\circ}(L_I)$ . Then, by (\*),  $\lambda$  can only have parts equal to 1 or  $e\ell^i$  for some  $i \geq 0$ . Let

$$m_{-1} := \text{multiplicity of } 1 \text{ as a part of } \lambda,$$

$$m_i := \text{multiplicity of } e\ell^i \text{ as a part of } \lambda \quad (i = 0, 1, 2, \dots).$$

(Here, we understand that  $m_{-1} = 0$  if  $e = 1$ .) We have  $\mathcal{N}(I, X) = \mathcal{N}(I)$  and  $X$  can be extended to  $\mathcal{N}(I)$ ; furthermore,  $\mathcal{W}(I, X)$  is a Coxeter group  $W_1$  isomorphic to

$$\mathfrak{S}_{m_{-1}} \times \mathfrak{S}_{m_0} \times \mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2} \times \cdots.$$

(Thus,  $\Omega = \{1\}$  in this case.) Finally, the parameter function  $\pi: W_1 \rightarrow k^{\times}$  is given as follows. We have  $\pi(s) = \bar{q}$  for a generator  $s$  in the  $\mathfrak{S}_{m_{-1}}$ -factor, and  $\pi(s) = 1$  otherwise. See Dipper [16, Part II, §4] for more details. The simple  $\mathcal{H}$ -modules are classified by Dipper–James [20].

**Example 3.8.** Let  $G = \mathrm{GU}_n(\mathbb{F}_q)$ . Then we have again

$$|\mathrm{Uch}_\ell(G)| = |\mathrm{Uch}(G)| = \text{number of partitions of } n.$$

Furthermore, we have

$$|\mathrm{Uch}_\ell^\circ(\mathrm{GU}_n(\mathbb{F}_q))| \leq p_2(n),$$

where  $p_2(n)$  denotes the number of all partitions of  $n$  with distinct parts; see Geck–Hiss–Malle [41, Prop. 6.2 and Prop. 6.8]. There are examples where equality holds (see case (b) below); in general, the exact number of cuspidal unipotent simple  $kG$ -modules is not known! Let us now consider the Hecke algebras arising in  $G$ .

Let us write  $n = m' + 2m$  where  $m, m' \in \mathbb{N}$ , and let  $\lambda$  be a partition of  $m$ . For any  $i \in \{1, \dots, m\}$ , let  $n_i \geq 0$  be the multiplicity of  $i$  as a part of  $\lambda$ . Then we have a corresponding Levi subgroup  $L_I \subseteq G$  such that

$$L_I \cong \mathrm{GU}_{m'}(\mathbb{F}_q) \times \prod_{i=1}^m \underbrace{(\mathrm{GL}_i(\mathbb{F}_{q^2}) \times \cdots \times \mathrm{GL}_i(\mathbb{F}_{q^2}))}_{n_i \text{ factors}};$$

furthermore, any Levi subgroup of  $G$  arises in this way (up to  $\approx$ ). Let us assume that  $X \in \mathrm{Uch}_\ell^\circ(L_I)$ . Then, by Example 3.7,  $\lambda$  can only have parts equal to 1,  $e$ ,  $e\ell$ ,  $e\ell^2$ ,  $\dots$ , where  $e$  is the multiplicative order of  $q^2$  modulo  $\ell$ . As in the previous example, let

$$m_{-1} := \text{multiplicity of } 1 \text{ as a part of } \lambda,$$

$$m_i := \text{multiplicity of } e\ell^i \text{ as a part of } \lambda \quad (i = 0, 1, 2, \dots).$$

(Here, we understand that  $m_{-1} = 0$  if  $e = 1$ .) By [42, Prop. 4.3], we have  $\mathcal{N}(I, X) = \mathcal{N}(I)$  and  $X$  can be extended to  $\mathcal{N}(I)$ ; furthermore,  $\mathcal{W}(I, X)$  is a Coxeter group  $W_1$  of type

$$B_{m_{-1}} \times B_{m_0} \times B_{m_1} \times B_{m_2} \times \dots$$

(Thus,  $\Omega = \{1\}$  in this case.) Finally, by [42, Prop. 4.4], the parameter function  $\pi: W_1 \rightarrow k^\times$  is given by

$$\begin{array}{ccccccc} B_{m_{-1}}: & p_1 & \bar{q}^2 & \bar{q}^2 & \dots & \bar{q}^2 & \bar{q}^2 \\ & \text{---} \text{---} \text{---} \text{---} \text{---} & \text{---} & \text{---} & \dots & \text{---} & \text{---} \\ B_{m_i}: & p'_1(i) & 1 & 1 & \dots & 1 & 1 \\ & \text{---} \text{---} \text{---} \text{---} \text{---} & \text{---} & \text{---} & \dots & \text{---} & \text{---} \end{array}$$

for  $i = 0, 1, 2, \dots$ , where  $p_1, p'_1(i) \in k^\times$  and  $\bar{q}$  denotes the image of  $q$  in  $k$ . The parameters  $p_1, p'_1(i)$  are only known in special cases. Let  $d$  be the multiplicative order of  $-q$  modulo  $\ell$ . Then

$$e = \begin{cases} d/2 & \text{if } d \text{ is even,} \\ d & \text{if } d \text{ is odd.} \end{cases}$$

The following distinction between odd and even values for  $d$  already occurs in the work of Fong–Srinivasan [26].

(a) *Assume that  $d = 2e$ . Then  $m' = t(t+1)/2$ ,  $p_1 = \bar{q}^{2t+1}$  and  $p'_1(i) \neq -1$ ; see [42, Lemma 4.9]. In this situation, the simple  $\mathcal{H}$ -modules are classified by Dipper–James [21] (see Example 6.9 below). A counting argument then yields that*

$$|\mathrm{Uch}_\ell^\circ(G)| = \begin{cases} 1 & \text{if } n = s(s+1)/2 \text{ for some } s \geq 1, \\ 0 & \text{otherwise;} \end{cases}$$

see [42, Theorem 4.11]. The unique  $X^\circ \in \mathrm{Uch}_\ell^\circ(G)$  (if it exists) can even be lifted to a cuspidal unipotent character in characteristic zero. The dimension of  $X^\circ$  is given by the polynomial in [12, p. 457]. This case is studied further by Gruber–Hiss [51].

(b) *Assume that  $d = 1$ , that is,  $\bar{q} = -1$ . Then  $p_1 = p'_1(i) = -1$ ; see [42, Lemma 4.6] and [50, Prop. 2.3.5]. In this situation, the simple  $\mathcal{H}$ -modules are again classified by Dipper–James [21] (see Example 6.9 below). A counting argument then yields that*

$$|\mathrm{Uch}_\ell^\circ(G)| = p_2(n);$$

see [42, Theorem 4.12]. The dimensions of these modules are not known.

(c) *Assume that  $d > 1$  is odd. Then  $p'_1(i) = -1$  for all  $i \geq 0$ ; see [50, Prop. 2.3.5]. As before, the simple modules of the Hecke algebra corresponding to the  $B_{m_i}$ -factor ( $i \geq 0$ ) are classified by Dipper–James [21]. The analogous problem for the factor of type  $B_{m_{-1}}$  has been solved by Ariki–Mathas [4] and Ariki [2], in terms of so-called Kleshchev bipartitions. These bipartitions are defined in a recursive way. Recently, Jacon [55] obtained a whole family of different parametrizations by so-called FLOTW bipartitions, which have the advantage of being defined in a non-recursive way; furthermore, they can be adapted more accurately to the values of the parameter function  $\pi$ . (We will discuss all this in more detail in Sections 7 and 8.)*

**Open problem:** *Determine  $p_1$  for  $d > 1$  odd. Or even better, find a general practical method for determining the function  $\pi: W_1 \rightarrow k^\times$ .*

Some partial results on the computation of  $\pi(s)$  are contained in [42, §3]; Hiss (unpublished) actually has a conjecture about  $p_1$ . Note that, in characteristic zero, the parameters  $\pi(s)$  are completely known; see Lusztig [68], Table II, p. 35. In this case, we have  $\pi(s) = q^{L(s)}$  for  $s \in S_1$ , where  $L: W_1 \rightarrow \mathbb{N}$  is a weight function such that  $L(s) > 0$  for all  $s \in S_1$ .

To summarize, the above results show that it would be very interesting to know a parametrization of the simple modules of an Iwahori–Hecke algebra  $H_k(W_1, \pi)$  where  $\pi: W_1 \rightarrow k^\times$  has values in an algebraically closed field  $k$  of characteristic  $\ell > 0$ . In particular, it would be interesting to know in which way the parametrization depends on the function  $\pi$ . In this context, we may assume that  $\pi(s) = \xi^{L(s)}$  for all  $s \in S_1$ , where  $\xi \in k^\times$  has finite order and  $L: W_1 \rightarrow \mathbb{N}$  is a weight function (see Remark 2.7).

## 4. GENERIC IWAHORI–HECKE ALGEBRAS AND SPECIALIZATIONS

We now consider a finite Coxeter group  $W$  with generating set  $S$  and a weight function  $L: W \rightarrow \mathbb{N}$ , without reference to a realization of  $W$  in the framework of groups with a BN-pair. Our aim is to develop the representation theory of the associated Iwahori–Hecke algebras over various fields  $k$ . One of first decisive observations is the fact that these algebras can be defined “generically” over a polynomial ring, where the parameters are powers of the indeterminate. The precise definitions are as follows. (A general reference is [47].)

Let  $A = \mathbb{Z}[v, v^{-1}]$  be the ring of Laurent polynomials in an indeterminate  $v$ . Then there exists an associative algebra  $\mathbf{H} = \mathbf{H}_A(W, L)$  over  $A$ , which is free as an  $A$ -module with basis  $\{T_w \mid w \in W\}$  such that the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ v^{2L(s)} T_{sw} + (v^{2L(s)} - 1) T_w & \text{if } l(sw) < l(w), \end{cases}$$

where  $s \in S$  and  $w \in W$ . The fact that the parameters are even powers of  $v$  will play a role in connection with the construction of the Kazhdan–Lusztig basis of  $W$  (to be discussed in Section 5) and also in connection with the question of splitting fields.

**4.1. Specialization.** Let  $k$  be a field and  $\xi \in k^\times$  be an element which has a square root in  $k^\times$ . Then there is a ring homomorphism  $\theta: A \rightarrow k$  such that  $\theta(v^2) = \xi$ . Considering  $k$  as an  $A$ -module via  $\theta$ , we set

$$\mathbf{H}_{k,\xi} := k \otimes_A \mathbf{H}.$$

Thus,  $\mathbf{H}_{k,\xi}$  is an associative  $k$ -algebra with a basis  $\{T_w \mid w \in W\}$  such that

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ \xi^{L(s)} T_{sw} + (\xi^{L(s)} - 1) T_w & \text{if } l(sw) < l(w), \end{cases}$$

where  $s \in S$  and  $w \in W$ . This shows that the endomorphism algebras arising in the context of Harish-Chandra series as in the previous sections are obtained via specialization from generic Iwahori–Hecke algebras.

In order to deal with non-semisimple specializations of  $\mathbf{H}$ , we shall need some fundamental results on the structure of  $\mathbf{H}_{K,v}$  where  $K$  is the field of fractions of  $A$  and  $\theta: A \hookrightarrow K$  is the inclusion. For technical simplicity, we will assume from now on that

*$W$  is a finite Weyl group,*

that is, the product of two generators of  $W$  is 2, 3, 4 or 6. Then it is known that every complex representation of  $W$  can be realized over  $\mathbb{Q}$ , that is, the group algebra  $\mathbb{Q}W$  is split semisimple (see [47, 6.3.8]). Using the specialization  $v \mapsto 1$  and Tits’ deformation argument, it is not hard to show that  $\mathbf{H}_{K',v}$  is split semisimple and abstractly isomorphic to the group algebra  $K'W$ , where  $K'$  is a sufficiently large finite extension of  $K$ ; see [47,

8.1.7]. We have the following more precise result, which combines work of Benson–Curtis, Lusztig, Digne–Michel; see [47, 9.3.5]:

**Theorem 4.2.** *The algebra  $\mathbf{H}_{k,v}$  is split semisimple and abstractly isomorphic to the group algebra  $KW$ .*

Now, an isomorphism  $\mathbf{H}_{K,v} \cong KW$  certainly induces a bijection between  $\text{Irr}(\mathbf{H}_{K,v})$  and  $\text{Irr}_{\mathbb{Q}}(W)$ . To describe this, it will be convenient to work with characters. As in the case of finite groups, the character of an  $\mathbf{H}_{K,v}$ -module  $V$  is the function  $\chi_V: \mathbf{H}_{K,v} \rightarrow K$  sending  $h \in \mathbf{H}_{K,v}$  to the trace of  $h$  acting on  $V$ . Since  $A$  is integrally closed in  $K$ , a general argument (see [47, 7.3.9]) shows that  $\chi_V(T_w) \in A$  for all  $w \in W$ . Let  $\mathbf{H}_{K,v}^{\wedge}$  be the set of irreducible characters of  $\mathbf{H}_{K,v}$ . Once Theorem 4.2 is established, *Tits' Deformation Theorem* (see [47, 7.4.6]) yields the following result.

**Theorem 4.3.** *There is a bijection  $W^{\wedge} \xrightarrow{\sim} \mathbf{H}_{K,v}^{\wedge}$ , denoted  $\chi \mapsto \chi_v$ , which is uniquely determined by the condition that*

$$\chi(w) = \chi_v(T_w)|_{v=1} \quad \text{for all } w \in W.$$

Explicit tables or combinatorial algorithms for the values  $\chi_v(T_w)$  (where  $w$  has minimal length in its conjugacy class in  $W$ ) are known for all types of  $W$  and all weight functions  $L$ ; this is one of the main themes of [47].

As before,  $\mathbf{H}$  is a symmetric algebra with trace function  $\tau: \mathbf{H} \rightarrow A$  given by  $\tau(T_1) = 1$  and  $\tau(T_w) = 0$  for  $w \neq 1$ . By extension of scalars, we obtain a trace function  $\tau_K: \mathbf{H}_{K,v} \rightarrow K$ . Since  $\mathbf{H}_{K,v}$  is split semisimple, we can write  $\tau_K$  as a linear combination of  $\mathbf{H}_{K,v}^{\wedge}$ , where all irreducible characters appear with a non-zero coefficient. (This is a general result about split semisimple symmetric algebras; see [47, 7.2.6].) Thus, we can write

$$\tau_K = \sum_{\chi \in W^{\wedge}} \mathbf{c}_{\chi}^{-1} \chi_v \quad \text{where} \quad \mathbf{c}_{\chi} \in K^{\times}.$$

Since  $A$  is integrally closed in  $K$ , we have  $\mathbf{c}_{\chi} \in A$  for all  $\chi \in W^{\wedge}$ . (Again, this follows by a general argument on symmetric algebras; see [47, 7.3.9].) The constants  $\mathbf{c}_{\chi}$  appear in the *orthogonality relations* for the irreducible characters of  $\mathbf{H}_{K,v}$ ; see [47, 7.2.4]. Given  $\chi, \chi' \in W^{\wedge}$ , we have

$$\sum_{w \in W} v^{-L(w)} \chi_v(T_w) \chi_v(T_{w^{-1}}) = \begin{cases} \chi(1) \mathbf{c}_{\chi} & \text{if } \chi = \chi', \\ 0 & \text{if } \chi \neq \chi'. \end{cases}$$

We have the following more precise statement about the form of  $\mathbf{c}_{\chi}$ ; see the historical remarks in [47, §10.7 and §11.6] for the origins of this result.

**Theorem 4.4.** *For any  $\chi \in W^{\wedge}$ , we have*

$$\mathbf{c}_{\chi} = f_{\chi} v^{-2\alpha_{\chi}} \times \text{a product of cyclotomic polynomials in } v,$$

where  $f_{\chi}$  is a positive integer and  $\alpha_{\chi} \in \mathbb{N}$ . If  $L(s) > 0$  for all  $s \in S$ , then  $f_{\chi}$  is divisible only by primes which are not good for  $W$ ; here, good primes are defined as in the remarks following Theorem 3.4.



The polynomials  $\mathbf{c}_\chi$  are explicitly known for all  $W, L$ ; see the appendices of [12] or [47]. We write  $\alpha_E = \alpha_\chi$  and  $f_E = f_\chi$  if  $E \in \text{Irr}_\mathbb{Q}(W)$  affords  $\chi$ .

**Definition 4.5.** Let  $p$  be a prime number. We say that  $p$  is  $L$ -good if  $p$  does not divide any of the numbers  $f_E$  for  $E \in \text{Irr}_\mathbb{Q}(W)$ . By Theorem 4.4, a good prime for  $W$  is  $L$ -good if  $L(s) > 0$  for all  $s \in S$ .

*Remark 4.6.* Consider the extreme case where  $L(s) = 0$  for all  $s \in S$ . Then  $\mathbf{H} = AW$  and  $\mathbf{c}_\chi = |W|/\chi(1)$  for all  $\chi \in W^\wedge$ . Consequently, we have  $f_\chi = 1$  and  $\alpha_\chi = 0$  for all  $\chi$ . The  $L$ -good primes are prime numbers which do not divide the order of  $W$ . Of course, this case does not give any new information. However, for  $W$  of type  $B_n$ , it is interesting to look at cases where  $L(s) = 0$  for some  $s \in S$ ; see Example 4.9 below.

**Example 4.7.** Let  $W = W(G_2)$  be the Weyl group of type  $G_2$ ; that is, we have  $W = \langle s, t \mid s^2 = t^2 = (st)^6 = 1 \rangle$ . Since  $s, t$  are not conjugate, we can take any  $a, b \in \mathbb{N}$  and obtain a weight function  $L: W \rightarrow \mathbb{N}$  such that  $L(s) = a$  and  $L(t) = b$ . We have

$$\text{Irr}_\mathbb{Q}(W) = \{\mathbf{1}, \varepsilon, \varepsilon_1, \varepsilon_2, E_\pm\}$$

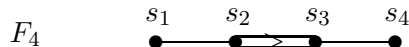
where  $\mathbf{1}$  is the unit representation,  $\varepsilon$  is the sign representation,  $\varepsilon_1, \varepsilon_2$  are two further 1-dimensional representations, and  $E_\pm$  are two 2-dimensional representations. By [47, 8.3.4], the polynomials  $\mathbf{c}_E$  are given by

$$\begin{aligned} \mathbf{c}_1 &= (v^{2a}+1)(v^{2b}+1)(v^{4a+4b}+v^{2a+2b}+1), & \mathbf{c}_\varepsilon &= v^{-6a-6b} \mathbf{c}_{\text{ind}}, \\ \mathbf{c}_{\varepsilon_1} &= v^{-6b}(v^{2a}+1)(v^{2b}+1)(v^{4a}+v^{2a+2b}+v^{4b}), & \mathbf{c}_{\varepsilon_2} &= v^{6b-6a} \mathbf{c}_{\varepsilon_1}, \\ \mathbf{c}_{E_\pm} &= 2v^{-2a-2b}(v^{2a+2b} \pm v^{a+b}+1)(v^{2a} \mp v^{a+b}+v^{2b}). \end{aligned}$$

This yields the following table:

$E$	$b > a > 0$		$b = a > 0$		$b > a = 0$	
	$f_E$	$\alpha_E$	$f_E$	$\alpha_E$	$f_E$	$\alpha_E$
$\mathbf{1}$	1	0	1	0	2	0
$\varepsilon$	1	$3b + 3a$	1	$6a$	2	$3b$
$\varepsilon_1$	1	$3b - 2a$	3	$a$	2	$3b$
$\varepsilon_2$	1	$a$	3	$a$	2	0
$E_+$	2	$b$	6	$a$	2	$b$
$E_-$	2	$b$	2	$a$	2	$b$

**Example 4.8.** Let  $W = W(F_4)$  be the Weyl group of type  $F_4$ , with generators and diagram given by:



A weight function  $L$  is specified by two integers  $a := L(s_1) = L(s_2) \geq 0$  and  $b := L(s_3) = L(s_4) \geq 0$ . There are 25 irreducible characters of  $W$ . The polynomials  $\mathbf{c}_E$  are listed in [47, p. 379]. One has to distinguish a number of cases in order to obtain the invariants  $\alpha_E$  and  $f_E$ ; see Table 1. (Note that these are the same cases that we found in [36].)

TABLE 1. The invariants  $f_\chi$  and  $\alpha_\chi$  for type  $F_4$ 

$E$	$b > 2a > 0$		$b = 2a > 0$		$2a > b > a > 0$		$b = a > 0$		$b > a = 0$	
	$f_E$	$\alpha_E$	$f_E$	$\alpha_E$	$f_E$	$\alpha_E$	$f_E$	$\alpha_E$	$f_E$	$\alpha_E$
$1_1$	1	0	1	0	1	0	1	0	6	0
$1_2$	1	$12b-9a$	2	$15a$	2	$11b-7a$	8	$4a$	6	$12b$
$1_3$	1	$3a$	2	$3a$	2	$-b+5a$	8	$4a$	6	0
$1_4$	1	$12b+12a$	1	$36a$	1	$12b+12a$	1	$24a$	6	$12b$
$2_1$	1	$3b-3a$	2	$3a$	2	$2b-a$	2	$a$	12	$3b$
$2_2$	1	$3b+9a$	2	$15a$	2	$2b+11a$	2	$13a$	12	$3b$
$2_3$	1	$a$	1	$a$	1	$a$	2	$a$	3	0
$2_4$	1	$12b+a$	1	$25a$	1	$12b+a$	2	$13a$	3	$12b$
$4_1$	2	$3b+a$	2	$7a$	2	$3b+a$	8	$4a$	6	$3b$
$9_1$	1	$2b-a$	2	$3a$	2	$b+a$	1	$2a$	2	$2b$
$9_2$	1	$6b-2a$	1	$10a$	1	$6b-2a$	8	$4a$	2	$6b$
$9_3$	1	$2b+2a$	1	$6a$	1	$2b+2a$	8	$4a$	2	$2b$
$9_4$	1	$6b+3a$	2	$15a$	2	$5b+5a$	1	$10a$	2	$6b$
$6_1$	3	$3b+a$	3	$7a$	3	$3b+a$	3	$4a$	12	$3b$
$6_2$	3	$3b+a$	3	$7a$	3	$3b+a$	12	$4a$	12	$3b$
$12_1$	3	$3b+a$	6	$7a$	6	$3b+a$	24	$4a$	6	$3b$
$4_2$	1	$b$	1	$2a$	1	$b$	2	$a$	6	$b$
$4_3$	1	$7b-3a$	1	$11a$	1	$7b-3a$	4	$4a$	6	$7b$
$4_4$	1	$b+3a$	1	$5a$	1	$b+3a$	4	$4a$	6	$b$
$4_5$	1	$7b+6a$	1	$20a$	1	$7b+6a$	2	$13a$	6	$7b$
$8_1$	1	$3b$	1	$6a$	1	$3b$	1	$3a$	12	$3b$
$8_2$	1	$3b+6a$	1	$12a$	1	$3b+6a$	1	$9a$	12	$3b$
$8_3$	1	$b+a$	2	$3a$	2	$3a$	1	$3a$	3	$b$
$8_4$	1	$7b+a$	2	$15a$	2	$6b+3a$	1	$9a$	3	$7b$
$16_1$	2	$3b+a$	2	$7a$	2	$3b+a$	4	$4a$	6	$3b$

**Example 4.9.** Let  $n \geq 2$  and  $W_n = W(B_n)$  be a Coxeter group of type  $B_n$  with generators and diagram given by:

$$B_n \quad \begin{array}{c} t \quad s_1 \quad s_2 \quad \dots \quad s_{n-1} \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \end{array}$$

A weight function  $L$  is specified by two integers  $b := L(t) \geq 0$  and  $a := L(s_1) = \dots = L(s_{n-1}) \geq 0$ . It is well-known that we have a parametrization

$$\text{Irr}_{\mathbb{Q}}(W_n) = \{E^{\lambda} \mid \lambda \vdash n\}.$$

Here, the notation  $\lambda \vdash n$  means that  $\lambda$  is a bipartition of  $n$ , that is, a pair of partitions  $\lambda = (\lambda_{(1)}, \lambda_{(2)})$  such that  $|\lambda_{(1)}| + |\lambda_{(2)}| = n$ . For example, the trivial representation is labelled by the pair  $((n), \emptyset)$  and the sign representation is labelled by  $(\emptyset, (1^n))$ ; see [47, §5.5]. The polynomials  $\mathbf{c}_{E^{\lambda}}$  are determined as follows. Let  $\lambda = (\lambda_{(1)}, \lambda_{(2)}) \vdash n$ . By adding zeros if necessary, we write

$\lambda_{(1)}$  and  $\lambda_{(2)}$  in the form

$$\begin{aligned}\lambda_{(1)} &= (\lambda_{(1),1} \geq \lambda_{(1),2} \geq \cdots \geq \lambda_{(1),m} \geq \lambda_{(1),m+1} \geq 0), \\ \lambda_{(2)} &= (\lambda_{(2),1} \geq \lambda_{(2),2} \geq \cdots \geq \lambda_{(2),m}),\end{aligned}$$

for some  $m \geq 0$ . Then we define the following symbol:

$$\Lambda = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m & \alpha_{m+1} \\ & \beta_1 & \beta_2 & \cdots & \beta_m \end{pmatrix}$$

where  $\alpha_i = i - 1 + \lambda_{(1),m+2-i}$  and  $\beta_i = i - 1 + \lambda_{(2),m+1-i}$  for all  $i \geq 1$ . Then

$$\begin{aligned}c_{E\lambda} &= \frac{v^{2am(2m+1)(m-2)/3} (v^{2a} + v^{2b})^m}{(v^{2a} - 1)^n \prod_{i=1}^{m+1} \prod_{j=1}^m (v^{2a(\alpha_i-1)+2b} + v^{2a\beta_j})} \\ &\quad \times \frac{\prod_{i=1}^{m+1} \prod_{k=1}^{\alpha_i} (v^{2ak} - 1)(v^{2a(k-1)+2b} + 1)}{\prod_{1 \leq i' < i \leq m+1} (v^{2a\alpha_i} - v^{2a\alpha_{i'}})} \\ &\quad \times \frac{\prod_{j=1}^m \prod_{k=1}^{\beta_j} (v^{2ak} - 1)(v^{2a(k+1)-2b} + 1)}{\prod_{1 \leq j' < j \leq m} (v^{2a\beta_j} - v^{2a\beta_{j'}})}.\end{aligned}$$

see [12, p. 447]. Lusztig [73, Prop. 22.14] has obtained explicit combinatorial formulae for the invariants  $f_\chi$  and  $\alpha_\chi$  (as a function of the parameters  $a, b$ ). The following special cases are worth mentioning.

• **The asymptotic case<sup>1</sup> in type  $B_n$ .** Assume that  $b > (n-1)a > 0$ . Then it is not hard to check (directly using the above formula) that

$$f_{E\lambda} = 1 \quad \text{and} \quad \alpha_{E\lambda} = b|\lambda_{(2)}| + a(n(\lambda_{(1)}) + 2n(\lambda_{(2)}) - n(\lambda_{(2)}^*))$$

where the star denotes the conjugate partition and where, for any partition  $\nu$ , we write  $n(\nu) := \sum_{i=1}^r (i-1)\nu_i$  for  $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r > 0)$ ; see [34, Remark 5.1].

• **Type  $A_{n-1}$ .** The parabolic subgroup generated by  $s_1, \dots, s_{n-1}$  is isomorphic to the symmetric group  $\mathfrak{S}_n$ , where  $s_i$  corresponds to the transposition  $(i, i+1)$ . For a partition  $\nu$  of  $n$ , denote by  $E^\nu$  the restriction of  $E^{(\nu, \emptyset)}$  to  $\mathfrak{S}_n$ . Then it is well-known that

$$\text{Irr}_{\mathbb{Q}}(\mathfrak{S}_n) = \{E^\nu \mid \nu \vdash n\}.$$

---

<sup>1</sup>This term comes from the work of Bonnafé–Iancu [6], where the Kazhdan–Lusztig cells in this case are described.

For example, the partition  $(n)$  labels the trivial representation and  $(1^n)$  labels the sign representation. Furthermore, the formulae from the “asymptotic case” give the correct values for  $f_E$  and  $\alpha_E$ , that is, for  $a > 0$ , we have  $f_{E^\nu} = 1$  and  $\alpha_{E^\nu} = n(\nu)a$  for all  $\nu \vdash n$ . See [47, §10.5].

• **Type  $D_n$ .** Assume that  $a > 0$  and  $L(t) = b = 0$ . We set  $s_0 := ts_1t$ . Then  $W'_n := \langle s_0, s_1, \dots, s_{n-1} \rangle$  is a Coxeter group of type  $D_n$ . For partitions  $\lambda, \mu$  such that  $(\lambda, \mu) \vdash n$ , we denote by  $E^{[\lambda, \mu]}$  the restriction of  $E^{(\lambda, \mu)}$  to  $W'_n$ . Then

$$\begin{aligned} E^{[\lambda, \mu]} &\cong E^{[\mu, \lambda]} \in \text{Irr}(W'_n) && \text{if } \lambda \neq \mu, \\ E^{[\lambda, \lambda]} &\cong E^{[\lambda, +]} \oplus E^{[\lambda, -]} && \text{if } \lambda = \mu \text{ (and } n \text{ even),} \end{aligned}$$

where  $E^{[\lambda, \pm]} \in \text{Irr}(W'_n)$  and  $E^{[\lambda, +]} \not\cong E^{[\lambda, -]}$ . This yields (see [47, Chap. 5] for more details):

$$\text{Irr}(W'_n) = \{E^{[\lambda, \mu]} \mid (\lambda, \mu) \vdash n, \lambda \neq \mu\} \cup \{E^{[\lambda, \pm]} \mid n \text{ even}, \lambda \vdash n/2\}.$$

Let  $L'$  be the restriction of  $L$  to  $W'_n$ . Then  $L'(s_i) = a$  for  $0 \leq i \leq n-1$ , that is,  $L'$  is just  $a$  times the length function on  $W'_n$ ; see, for example, [47, Lemma 1.4.12]. By [47, Prop. 10.5.6], we have

$$\begin{aligned} \alpha_{E^{[\lambda, \mu]}} &= \alpha_{E^{(\lambda, \mu)}} \quad \text{and} \quad f_{E^{[\lambda, \mu]}} = f_{E^{(\lambda, \mu)}}, && \text{if } \lambda \neq \mu, \\ \alpha_{E^{[\lambda, \pm]}} &= \alpha_{E^{(\lambda, \lambda)}} \quad \text{and} \quad f_{E^{[\lambda, \pm]}} = 2f_{E^{(\lambda, \lambda)}}, && \text{if } n \text{ even and } \lambda \vdash n/2. \end{aligned}$$

• Another extreme case is given by  $a = 0$  and  $b > 0$ . Using the above formula for  $\mathbf{c}_{E^\lambda}$ , one easily checks that  $\alpha_{E^\lambda} = |\lambda_{(2)}|b$  for all  $\lambda \vdash n$ .

**4.10. Modular decomposition numbers.** Let  $k$  be a field and  $\theta: A \rightarrow k$  be a ring homomorphism. Let  $\xi = \theta(v^2)$  and consider the specialized algebra  $\mathbf{H}_{k, \xi}$ . Given  $E \in \text{Irr}_{\mathbb{Q}}(W)$  and  $M \in \text{Irr}(\mathbf{H}_{k, \xi})$ , we would like to define a *decomposition number*  $[E : M]$ , as in Brauer’s modular representation theory of finite groups. Here, some care is needed since we are not necessarily working with discrete valuation rings. So let  $\chi \in W^\wedge$  be the character of  $E$  and  $\chi_v$  be the corresponding irreducible character of  $\mathbf{H}_{K, v}$ . Let

$$\rho_v: \mathbf{H}_{K, v} \rightarrow M_d(K), \quad T_w \mapsto (a_{ij}(T_w)),$$

be a matrix representation affording  $\chi_v$ , where  $d = \dim E$ . Now,  $\mathfrak{p} = \ker(\theta)$  is a prime ideal in  $A$  and the localization  $A_{\mathfrak{p}}$  is a regular local ring of Krull dimension  $\leq 2$ ; see Matsumura [77] for these notions. Hence, by Du–Parshall–Scott [24, 1.1.1], we can assume that  $\rho_v$  satisfies the condition

$$\rho_v(T_w) \in M_d(A_{\mathfrak{p}}) \quad \text{for all } w \in W.$$

Now,  $\theta$  certainly extends to a ring homomorphism  $\theta_{\mathfrak{p}}: A_{\mathfrak{p}} \rightarrow k$ . Applying  $\theta_{\mathfrak{p}}$ , we obtain a representation

$$\rho_{k, \xi}: \mathbf{H}_{k, \xi} \rightarrow M_d(k), \quad T_w \mapsto (\theta_{\mathfrak{p}}(a_{ij}(T_w))).$$

This representation may no longer be irreducible. For any  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ , let  $[E : M]$  be the multiplicity of  $M$  as a composition factor of the  $\mathbf{H}_{k,\xi}$ -module affording  $\rho_{k,\xi}$ . There are some choices involved in this process, but one can show that  $[E : M]$  is independent of these choices; see [24, 1.1.2]. Thus, we obtain a well-defined matrix

$$D = ([E : M])_{E \in \text{Irr}_{\mathbb{Q}}(W), M \in \text{Irr}(\mathbf{H}_{k,\xi})}$$

which is called the *decomposition matrix* associated with  $\theta$ . (One can also define  $D$  without using properties of regular local rings, but then some mild hypotheses on the ground field  $k$  are required; see [47, §7.4].)

**Theorem 4.11.** *In the above setting, assume that  $\theta(\mathbf{c}_{\chi}) \neq 0$  for all  $\chi \in W^{\wedge}$ . Then  $\mathbf{H}_{k,\xi}$  is split semisimple and, up to reordering the rows and columns,  $D$  is the identity matrix.*

The fact that  $\mathbf{H}_{k,\xi}$  is split semisimple is proved by the argument in [47, Cor. 9.3.9]. Once this is established, we can apply *Tits' Deformation Theorem* (see [47, 7.4.6]) and this yields the statement concerning  $D$ .

If the hypotheses of Theorem 4.11 are not satisfied, then our aim is to find a good parametrization of  $\text{Irr}(\mathbf{H}_{k,\xi})$  using properties of  $D$ . The following example provides a model of what we are looking for.

**Example 4.12.** Let  $W = \mathfrak{S}_n$  be the symmetric group, with generators  $\{s_1, \dots, s_{n-1}\}$  as usual. Since all  $s_i$  are conjugate, we are in the “equal parameter case” and so  $L(s_i) = a$  for all  $i$ . Let us assume that  $a > 0$ . We set

$$e := \min\{i \geq 2 \mid 1 + \xi^a + \xi^{2a} + \dots + \xi^{(i-1)a} = 0\}.$$

(If no such  $i$  exists, we set  $e = \infty$ .) The following results are due to Dipper and James [19]. For any partition  $\lambda \vdash n$ , we have a corresponding *Specht module*  $S_k^{\lambda} \in \mathbf{H}_{k,\xi}\text{-mod}$ . This module has the property that

$$[E^{\lambda} : M] = \text{multiplicity of } M \text{ as a composition factor of } S_k^{\lambda},$$

for any  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ . (Actually, we shall denote by  $S_k^{\lambda}$  the module that is labelled by  $\lambda^*$  in [19], where the star denotes the conjugate partition.) Furthermore, there is an  $\mathbf{H}_{k,\xi}$ -equivariant symmetric bilinear form on  $S_k^{\lambda}$ ; then  $\text{rad}(S_k^{\lambda})$ , the radical of that form, is an  $\mathbf{H}_{k,\xi}$ -submodule of  $S_k^{\lambda}$ . We set

$$D^{\lambda} := S_k^{\lambda} / \text{rad}(S_k^{\lambda}) \in \mathbf{H}_{k,\xi}\text{-mod}.$$

Let  $\Lambda_n^{\circ} := \{\lambda \vdash n \mid D^{\lambda} \neq \{0\}\}$ . Then we have

$$\text{Irr}(\mathbf{H}_{k,\xi}) = \{D^{\lambda} \mid \lambda \in \Lambda_n^{\circ}\} \quad \text{and} \quad \Lambda_n^{\circ} = \{\lambda \vdash n \mid \lambda \text{ is } e\text{-regular}\}.$$

(A partition is  $e$ -regular if no part is repeated  $e$  or more times.) Thus, we have a “natural” parametrization of the simple  $\mathbf{H}_{k,\xi}$ -modules by a subset of the indexing set for  $\text{Irr}_{\mathbb{Q}}(\mathfrak{S}_n)$ . We wish to recover that parametrization from properties of the decomposition matrix  $D$ . For this purpose, we need

to introduce the dominance order on partitions. Let  $\lambda, \mu$  be partitions of  $n$ , with parts

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0), \quad \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0).$$

We write  $\lambda \leq \mu$  if  $\sum_{i=1}^j \lambda_i \leq \sum_{i=1}^j \mu_i$  for all  $j$ . Then we have:

$$(*) \quad \begin{cases} [E^\mu : D^\mu] = 1 & \text{for any } \mu \in \Lambda_n^\circ, \\ [E^\lambda : D^\mu] \neq 0 & \Rightarrow \lambda \leq \mu, \end{cases}$$

see [19, Theorem 7.6]. Note that the above conditions uniquely determine the set  $\Lambda_n^\circ$ . Indeed, let  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ . Then  $(*)$  shows that the set

$$\{\lambda \vdash n \mid [E^\lambda : M] \neq 0\}$$

has a unique maximal element with respect to  $\leq$ , namely, the unique  $\mu \in \Lambda_n^\circ$  such that  $M = D^\mu$ . Finally, let us consider the invariants  $\alpha_{E^\lambda}$ . By Example 4.9, we have  $\alpha_{E^\lambda} = n(\lambda)a$ . Now it is known that, for any  $\nu, \nu' \vdash n$ , we have  $\nu \leq \nu' \Rightarrow n(\nu') \leq n(\nu)$ , with equality only if  $\nu = \nu'$  (see, for example, [47, Exc. 5.6]). Thus, given  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$  and setting

$$\check{\alpha}_M := \min\{\alpha_{E^\lambda} \mid \lambda \vdash n \text{ and } [E^\lambda : M] \neq 0\},$$

there is a unique  $\mu \in \Lambda_n^\circ$  such that  $\check{\alpha}_M = \alpha_{E^\mu}$  and  $[E^\mu : M] \neq 0$ . Hence, the set  $\Lambda_n^\circ$  can be characterized as follows:

$$\Lambda_n^\circ = \{\lambda \vdash n \mid \exists M \in \text{Irr}(\mathbf{H}_{k,\xi}) \text{ such that } [E^\lambda : M] \neq 0, \check{\alpha}_M = \alpha_{E^\lambda}\}.$$

Note that this characterization does not require the explicit knowledge of all decomposition numbers.

We can now formalize the above discussion as follows.

**Definition 4.13.** Let  $k$  be a field and  $\theta: A \rightarrow k$  be a ring homomorphism; let  $\xi = \theta(v^2)$ . For any  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ , we define

$$\check{\alpha}_M := \min\{\alpha_E \mid E \in \text{Irr}_{\mathbb{Q}}(W) \text{ and } [E : M] \neq 0\}.$$

We say that a subset  $\mathcal{B}_{k,\xi} \subseteq \text{Irr}_{\mathbb{Q}}(W)$  is a “canonical basic set” for  $\mathbf{H}_{k,\xi}$  if the following conditions are satisfied:

- (a) There is a bijection  $\text{Irr}(\mathbf{H}_{k,\xi}) \xrightarrow{\sim} \mathcal{B}_{k,\xi}$ , denoted  $M \mapsto E(M)$ , such that  $[E(M) : M] = 1$  and  $\alpha_{E(M)} = \check{\alpha}_M$ .
- (b) Given  $E \in \text{Irr}_{\mathbb{Q}}(W)$  and  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ , we have

$$[E : M] \neq 0 \quad \Rightarrow \quad \check{\alpha}_M < \alpha_E \quad \text{or} \quad E = E(M).$$

Note that, if a canonical basic set exists, then (a) and (b) uniquely determine the set  $\mathcal{B}_{k,\xi} \subseteq \text{Irr}_{\mathbb{Q}}(W)$  and the bijection  $M \mapsto E(M)$ . If  $\mathcal{B}_{k,\xi}$  exists, then the submatrix

$$D^\circ = ([E(M) : M'])_{M, M' \in \text{Irr}(\mathbf{H}_{k,\xi})}$$

is square and lower triangular with 1 on the diagonal, when we order the modules in  $\text{Irr}(\mathbf{H}_{k,\xi})$  according to increasing values of  $\check{\alpha}_M$ . More precisely, we have a block lower triangular shape

$$D^\circ = \begin{pmatrix} D_0^\circ & & 0 \\ & D_1^\circ & \\ & & \ddots \\ * & & & D_N^\circ \end{pmatrix},$$

where the block  $D_i^\circ$  has rows and columns labelled by those  $E(M)$  and  $M'$ , respectively, where  $\check{\alpha}_M = \check{\alpha}_{M'} = i$ , and each  $D_i^\circ$  is the identity matrix.

*Remark 4.14.* Assume that  $\theta(\mathbf{c}_\chi) \neq 0$  for all  $\chi \in W^\wedge$ . Then, by Theorem 4.11,  $D$  is the identity matrix and so  $\mathcal{B}_{k,\xi} = \text{Irr}_{\mathbb{Q}}(W)$  is the unique canonical basic set. Hence, the interesting cases are those where  $\theta(\mathbf{c}_\chi) = 0$  for some  $\chi$ . By Theorem 4.4, this implies that the characteristic of  $k$  must be a prime which is not  $L$ -good, or that  $\theta(v)$  must be a root of unity in  $k$ .

The following example shows that canonical basic sets do not always exist.

**Example 4.15.** Let us consider the Iwahori–Hecke algebra of type  $G_2$  as in Example 4.7. Let  $L$  be the weight function given by length, let  $k$  be a field of characteristic 2 and  $\theta: A \rightarrow k$  be a specialization such that  $\xi = \theta(v^2) = 1$ . Using the explicit matrix representations in [47, §8.6], we find that the decomposition matrix is given by

$E$	$\alpha_E$	$[E : M]$	
$\mathbf{1}$	0	1	0
$\varepsilon_1$	1	1	0
$\varepsilon_2$	1	1	0
$\varepsilon$	6	1	0
$E_+$	1	0	1
$E_-$	1	0	1

Thus, there is no subset  $\mathcal{B}_{k,1}$  satisfying the conditions in Definition 4.13. Note that 2 is not  $L$ -good in this case.

In Section 6, we will show by a general method, following Geck [29], [32] and Geck–Rouquier [48], that canonical basic sets do exist when the characteristic of  $k$  is zero or a prime which is  $L$ -good. This general method relies on some properties of the Kazhdan–Lusztig basis of  $\mathbf{H}$ , which are not yet known to hold in general if  $L$  is not constant on  $S$ . But note that, as far as unequal parameters are concerned, we only have to deal with groups of type  $G_2$ ,  $F_4$  and  $B_n$  (any  $n \geq 2$ ). By an explicit (and easy) computation, one can determine the decomposition matrices in type  $G_2$  for all choices of  $L$  and  $\theta: A \rightarrow k$  (using the matrix representations in [47, §8.6]). By inspection, one finds that there is a canonical basic set  $\mathcal{B}_{k,\xi}$  if the characteristic of  $k$  is zero or an  $L$ -good prime. A similar statement can be verified in type  $F_4$ , using the decomposition matrices computed by Geck–Lux [46], Bremke [8]

and McDonough–Pallikaros [78]. Thus, the remaining case is type  $B_n$ , and this will be discussed in Example 6.9 and Section 8.

## 5. THE KAZHDAN–LUSZTIG BASIS AND THE $\mathbf{a}$ -FUNCTION

We keep the setting of the previous section, where  $W$  is a finite Weyl group and  $L: W \rightarrow \mathbb{N}$  is a weight function. Now we introduce the Kazhdan–Lusztig basis of  $\mathbf{H}$  and explain some theoretical constructions arising from it, most notably Lusztig’s ring  $\mathbf{J}$ . We will illustrate the power of these methods by giving new and conceptual proofs for some of the results in Section 4 on the structure of  $\mathbf{H}_{K,v}$ . In the following section, we shall discuss applications to non-semisimple specializations of  $\mathbf{H}$ .

Let us assume that  $L(s) > 0$  for all  $s \in S$ . It will be convenient to rescale the basis elements of  $\mathbf{H}$  as follows:

$$\tilde{T}_w := v^{-L(w)} T_w \quad \text{for any } w \in W.$$

Then the multiplication formulae read:

$$\tilde{T}_s \tilde{T}_w = \begin{cases} \tilde{T}_{sw} & \text{if } l(sw) > l(w), \\ \tilde{T}_{sw} + (v^{L(s)} - v^{-L(s)}) \tilde{T}_w & \text{if } l(sw) < l(w), \end{cases}$$

where  $s \in S$  and  $w \in W$ . To define the Kazhdan–Lusztig basis of  $\mathbf{H}$ , we need some ring homomorphisms of  $\mathbf{H}$ . First of all, we have a ring homomorphism  $A \rightarrow A$ ,  $f \mapsto \bar{f}$ , where  $\bar{f}$  is obtained from  $f$  by substituting  $v \mapsto v^{-1}$ . This extends to a ring homomorphism  $j: \mathbf{H} \rightarrow \mathbf{H}$  such that

$$j: \sum_{w \in W} a_w \tilde{T}_w \mapsto \sum_{w \in W} \bar{a}_w (-1)^{l(w)} \tilde{T}_w;$$

we have  $j^2 = \text{id}_{\mathbf{H}}$ . Next, define an  $A$ -linear map  $\mathbf{H} \rightarrow \mathbf{H}$ ,  $h \mapsto h^\dagger$ , by

$$\tilde{T}^\dagger := (-1)^{l(w)} \tilde{T}_{w^{-1}}^{-1} \quad (w \in W).$$

Then  $h \mapsto h^\dagger$  is an  $A$ -algebra automorphism whose square is the identity; furthermore,  $\dagger$  and  $j$  commute with each other. Hence we obtain a ring involution  $h \mapsto \bar{h}$  of  $\mathbf{H}$  by composing  $j$  and  $\dagger$ , that is, we have

$$\overline{\sum_{w \in W} a_w \tilde{T}_w} := \sum_{w \in W} \bar{a}_w \tilde{T}_{w^{-1}}^{-1}.$$

**Theorem 5.1** (Kazhdan–Lusztig; see Lusztig [73]). *For each  $w \in W$ , there exists a unique  $c_w \in \mathbf{H}$  such that*

$$\bar{c}_w = c_w \quad \text{and} \quad c_w \equiv \tilde{T}_w \pmod{\mathbf{H}_{<0}},$$

where  $\mathbf{H}_{<0}$  denotes the set of all  $v^{-1}\mathbb{Z}[v^{-1}]$ -linear combinations of basis elements  $\mathbf{T}_y$  ( $y \in W$ ). The elements  $\{c_w \mid w \in W\}$  form an  $A$ -basis of  $\mathbf{H}$ .

For example, we have  $c_1 = \tilde{T}_1$  and  $c_s = \tilde{T}_s + v^{-L(s)} \tilde{T}_1$  for  $s \in S$ .



Now the key to understanding the representations of a specialized algebra  $\mathbf{H}_{k,\xi}$  is the construction of Lusztig's ring  $\mathbf{J}$ . This involves the following ingredients. Since  $\{c_w\}$  is an  $A$ -basis of  $\mathbf{H}$ , we can write

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z \quad \text{where } h_{x,y,z} \in A;$$

we have  $\bar{h}_{x,y,z} = h_{x,y,z}$  for all  $x, y, z \in W$ . We define a function  $\mathbf{a}: W \rightarrow \mathbb{N}$  as follows. Let  $z \in W$ . Then we set

$$\mathbf{a}(z) := \min\{i \geq 0 \mid v^i h_{x,y,z} \in \mathbb{Z}[v] \text{ for all } x, y \in W\}.$$

It is easy to see that  $\mathbf{a}(1) = 0$  and that  $\mathbf{a}(z) = \mathbf{a}(z^{-1})$  for all  $z \in W$ . For  $x, y, z \in W$ , we set

$$\gamma_{x,y,z} := \text{constant term of } v^{\mathbf{a}(z)} h_{x,y,z^{-1}} \in \mathbb{Z}[v].$$

Thus, we obtain a family of integers  $\{\gamma_{x,y,z} \mid x, y, z \in W\} \subseteq \mathbb{Z}$  and we can try to use them to define a ring. Let  $\mathbf{J}$  be the free abelian group with basis  $\{t_w \mid w \in W\}$ . We define a bilinear product on  $\mathbf{J}$  by

$$t_x \cdot t_y := \sum_{z \in W} \gamma_{x,y,z} t_{z^{-1}} \quad (x, y \in W).$$

We would like to show that  $\mathbf{J}$  is an associative ring with an identity element. For this purpose, we need some further (and rather subtle) properties of the Kazhdan–Lusztig basis of  $\mathbf{H}$ .

**5.2. Lusztig's conjectures.** Let  $z \in W$  and consider  $\tau(c_z) \in A$ . By Theorem 5.1 and [73, Prop. 5.4], we can write

$$\tau(c_z) = n_z v^{-\Delta(z)} + \text{combination of smaller powers of } v,$$

where  $n_z \in \mathbb{Z} \setminus \{0\}$  and  $\Delta(z) \in \mathbb{N}$ . We set

$$\mathcal{D} := \{z \in W \mid \mathbf{a}(z) = \Delta(z)\}.$$

One easily checks that  $1 \in \mathcal{D}$  and that  $\mathcal{D}^{-1} = \mathcal{D}$ . Now Lusztig [73, Chap. 14] has formulated 15 properties P1–P15 of the above objects ( $h_{x,y,z}$ ,  $\gamma_{x,y,z}$ ,  $\mathcal{D}$  etc.) and conjectured that they always hold. For our purposes here, we only need the following ones:

- P2.** If  $d \in \mathcal{D}$  and  $x, y \in W$  satisfy  $\gamma_{x,y,d} \neq 0$ , then  $x = y^{-1}$ .
- P3.** If  $y \in W$ , there exists a unique  $d \in \mathcal{D}$  such that  $\gamma_{y^{-1},y,d} \neq 0$ .
- P4.** If  $x, y, z \in W$  satisfy  $h_{x,y,z} \neq 0$ , then  $\mathbf{a}(z) \geq \mathbf{a}(x)$  and  $\mathbf{a}(z) \geq \mathbf{a}(y)$ .
- P5.** If  $d \in \mathcal{D}$ ,  $y \in W$ ,  $\gamma_{y^{-1},y,d} \neq 0$ , then  $\gamma_{y^{-1},y,d} = n_d = \pm 1$ .
- P6.** If  $d \in \mathcal{D}$ , then  $d^2 = 1$ .
- P7.** For any  $x, y, z \in W$ , we have  $\gamma_{x,y,z} = \gamma_{y,z,x}$ .
- P8.** Let  $x, y, z \in W$  be such that  $\gamma_{x,y,z} \neq 0$ . Then  $\mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$ .
- P15'.** If  $x, x', y, w \in W$  satisfy  $\mathbf{a}(w) = \mathbf{a}(y)$ , then

$$\sum_{u \in W} h_{x,u,y} \gamma_{w,x',u^{-1}} = \sum_{u \in W} h_{x,w,u} \gamma_{u,x',y^{-1}}.$$

The above properties are known to hold in the following situations:

- (I) Recall that we are assuming that  $W$  is a finite Weyl group. Let us also assume that the weight function  $L$  is constant on  $S$  (the “equal parameter case”). Then, thanks to a deep geometric interpretation of the basis  $\{c_w\}$ , one can show that P1–P15 hold; see Lusztig [73, Chap. 15] and the references there. In the case where  $W = \mathfrak{S}_n$  is the symmetric group, elementary proofs for P1–P15 (that is, without reference to a geometric interpretation) can be found in [38].
- (II) Assume that  $W$  is of type  $B_n$  with diagram and weight function given as follows:

$$\begin{array}{ccccccc}
 B_n & & t & s_1 & s_2 & \dots & s_{n-1} \\
 & & \bullet & \bullet & \bullet & & \bullet \\
 L : & & b & a & a & & a
 \end{array}$$

where  $a, b \in \mathbb{N}$ . If  $b > (n-1)a > 0$  (the “asymptotic case” as in Example 4.9), then P1–P14 and P15’ are known to hold by Bonnafé–Iancu [6], Bonnafé [5], Geck [37] and Geck–Iancu [43].

If  $b = 0$  and  $a > 0$  (the case relevant for type  $D_n$ ), then P1–P15 are known to hold by a reduction argument to case (I); this is already due to Lusztig (see [32, §2] and the references there).

Partial results for type  $F_4$  with unequal parameters are contained in [36].

From now on, we will assume that P2–P8 and P15’ hold for  $W, L$ .

**Theorem 5.3** (Lusztig [73, 18.3, 18.9]).  *$\mathbf{J}$  is an associative ring with identity  $1_{\mathbf{J}} = \sum_{d \in \mathcal{D}} n_d t_d$ . Furthermore, setting  $\mathbf{J}_A = A \otimes_{\mathbb{Z}} \mathbf{J}$ , the  $A$ -linear map  $\phi: \mathbf{H} \rightarrow \mathbf{J}_A$  defined by*

$$\phi(c_w^\dagger) = \sum_{\substack{z \in W, d \in \mathcal{D} \\ \mathbf{a}(z) = \mathbf{a}(d)}} h_{w,d,z} \hat{n}_z t_z \quad (w \in W)$$

*is a homomorphism preserving the identity elements. Here, we set  $\hat{n}_z := n_d$  where  $d \in \mathcal{D}$  is the unique element such that  $\gamma_{z,z^{-1},d} \neq 0$ .*

Now let  $a \geq 0$  and consider the following  $A$ -submodules of  $\mathbf{H}$ :

$$\mathbf{H}^{\geq a} := \langle c_w^\dagger \mid \mathbf{a}(w) \geq a \rangle_A \quad \text{and} \quad \mathbf{H}^{>a} := \langle c_w^\dagger \mid \mathbf{a}(w) > a \rangle_A.$$

Then, by P4, both  $\mathbf{H}^{\geq a}$  and  $\mathbf{H}^{>a}$  are two-sided ideals of  $\mathbf{H}$ . Hence

$$\mathbf{H}^a := \mathbf{H}^{\geq a} / \mathbf{H}^{>a} = \langle [c_w^\dagger] \mid \mathbf{a}(w) = a \rangle_A$$

is an  $(\mathbf{H}, \mathbf{H})$ -bimodule. We define an  $A$ -bilinear map  $\mathbf{J}_A \times \mathbf{H}^a \rightarrow \mathbf{H}^a$  by

$$t_x \star [c_w^\dagger] := \sum_{z \in W} \gamma_{x,w,z^{-1}} \hat{n}_w \hat{n}_z [c_z^\dagger]$$

where  $x \in W$  and  $w \in W$  is such that  $\mathbf{a}(w) = a$ .

**Theorem 5.4** (Lusztig [73, 18.10]).  *$\mathbf{H}^a$  is a  $(\mathbf{J}_A, \mathbf{H})$ -bimodule and we have*

$$h \cdot [c_w^\dagger] = \phi(h) \star [c_w^\dagger] \quad \text{for all } h \in \mathbf{H} \text{ and } w \in W, \mathbf{a}(w) = a.$$

We invite the reader to check that the above two theorems indeed are proved by purely algebraic arguments, using only P2–P8 and P15’.

Now consider a ring homomorphism  $\theta: A \rightarrow k$  where  $k$  is a field. Since all of the above constructions are defined over  $A$ , we can extend scalars from  $A$  to  $k$  and obtain:

- the specialized algebras  $\mathbf{H}_{k,\xi}$  (where  $\xi = \theta(v^2)$ ) and  $\mathbf{J}_k = k \otimes_A J_A$ ;
- the induced  $k$ -algebra homomorphism  $\phi_{k,\xi}: \mathbf{H}_{k,\xi} \rightarrow \mathbf{J}_k$ ;
- a  $(\mathbf{J}_k, \mathbf{H}_{k,\xi})$ -bimodule structure on  $\mathbf{H}_{k,\xi}^a$  such that

$$h \cdot [c_w^\dagger] = \phi_{k,\xi}(h) \star [c_w^\dagger]$$

for all  $h \in \mathbf{H}_{k,\xi}$  and all  $w \in W$  such that  $\mathbf{a}(w) = a$ .

We are now in a position to obtain a number of applications.

**Proposition 5.5** (Lusztig [73, 18.12]). *In the above setting, the kernel of  $\phi_{k,\xi}$  is a nilpotent ideal and, hence, contained in the Jacobson radical of  $\mathbf{H}_{k,\xi}$ . Consequently,  $\phi_{k,\xi}$  is an isomorphism if  $\mathbf{H}_{k,\xi}$  is semisimple.*

*Proof.* To illustrate the use of the above constructions, we repeat Lusztig’s proof here. Let  $h \in \ker(\phi_{k,\xi})$  and  $a \geq 0$ . Then  $h \cdot [c_w^\dagger] = \phi_{k,\xi}(h) \star [c_w^\dagger] = 0$  for all  $w \in W$  such that  $\mathbf{a}(w) = a$ . In other words, this means that

$$h \mathbf{H}_{k,\xi}^{\geq a} \subseteq \mathbf{H}_{k,\xi}^{\geq a+1} \quad \text{for } a = 0, 1, 2, \dots$$

Now let  $N := \max\{\mathbf{a}(z) \mid z \in W\}$ . Repeating the above argument  $N + 1$  times, we obtain

$$h_1 \cdots h_{N+1} = h_1 \cdots h_{N+1} c_1^\dagger \in \mathbf{H}_k^{\geq N+1} = \{0\}$$

for any  $h_1, \dots, h_{N+1} \in \ker(\phi_{k,\xi})$ . Hence,  $\ker(\phi_{k,\xi})$  is a nilpotent ideal.  $\square$

The following examples are taken from Lusztig [73, Chap. 20].

**Example 5.6.** Consider the specialization  $\theta: A \rightarrow \mathbb{Q}$  such that  $\theta(v) = 1$ . Then  $\mathbf{H}_{\mathbb{Q},1} = \mathbb{Q}W$  is the group algebra of  $W$  and  $J_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} J$  is the algebra obtained by extension of scalars from  $\mathbb{Z}$  to  $\mathbb{Q}$ . Since  $\mathbb{Q}W$  is semisimple, Proposition 5.5 shows that

$$\phi_{\mathbb{Q},1}: \mathbb{Q}W \rightarrow J_{\mathbb{Q}}$$

is an isomorphism. In particular,  $J_{\mathbb{Q}}$  is split semisimple.

More generally, let  $k$  be a field whose characteristic is either zero or a prime which does not divide the order of  $W$ . Consider the specialization  $\theta: A \rightarrow k$  such that  $\theta(v) = 1$ . Then, again, we have  $\mathbf{H}_{k,1} = kW$  and this is a split semisimple algebra. Hence, as above,  $\phi_{k,1}: kW \rightarrow \mathbf{J}_k$  is an isomorphism.

**Example 5.7.** Let  $K = \mathbb{Q}(v)$ , the field of fractions of  $A$ , and consider the specialization  $\theta: A \rightarrow K$  given by inclusion. Now write

$$\phi_{K,v}(\tilde{T}_y) = \sum_{x \in W} b_{x,y} t_x \quad \text{where} \quad b_{x,y} \in A.$$

Explicitly, the coefficients  $b_{x,y}$  are obtained by writing  $\tilde{T}_y$  as an  $A$ -linear combination of  $c_w^\dagger$  and then to use the defining formula of  $\phi$ . Thus,  $B = (b_{x,y})_{x,y \in W}$  is a matrix with entries in  $A$ . If we set  $v = 1$ , we obtain the matrix of the homomorphism  $\phi_{\mathbb{Q},1}: \mathbb{Q}W \rightarrow J_{\mathbb{Q}}$  considered in the previous example. Since  $\phi_{\mathbb{Q},1}$  is an isomorphism, the determinant of the matrix  $B$  is a Laurent polynomial whose value at  $v = 1$  is non-zero. In particular,  $\det B \neq 0$  and so  $\phi_{K,v}$  is an isomorphism. Hence we obtain isomorphisms

$$\mathbf{H}_{K,v} \xrightarrow{\phi_{K,v}} \mathbf{J}_K \xrightarrow{\phi_{K,1}^{-1}} KW;$$

in particular,  $\mathbf{H}_{K,v}$  is split semisimple and isomorphic to  $KW$ . Thus, using Lusztig's ring  $J$ , we have recovered Theorem 4.2 by a general argument (assuming that P2–P8 and P15' hold).

Now, via the algebra isomorphisms constructed above, we may identify the following sets of simple modules:

$$\boxed{\text{Irr}_{\mathbb{Q}}(W) = \text{Irr}(\mathbf{J}_{\mathbb{Q}}) = \text{Irr}(\mathbf{J}_K) = \text{Irr}(\mathbf{H}_{K,v})}$$

where the second equality is given by extension of scalars from  $\mathbb{Q}$  to  $K$ . We shall denote these correspondences as follows. Let  $E \in \text{Irr}_{\mathbb{Q}}(W)$ . Composing the action of  $W$  on  $E$  with the inverse of the isomorphism  $\mathbb{Q}W \xrightarrow{\sim} \mathbf{J}_{\mathbb{Q}}$  in Example 5.6, we obtain a simple  $\mathbf{J}_{\mathbb{Q}}$ -module denoted by  $E_{\spadesuit}$ . Extending scalars from  $\mathbb{Q}$  to  $K$ , we obtain  $E_{\spadesuit,K} \in \text{Irr}(\mathbf{J}_K)$ . Finally, composing the action of  $\mathbf{J}_K$  on  $E_{\spadesuit,K}$  with the isomorphism  $\mathbf{H}_{K,v} \xrightarrow{\sim} \mathbf{J}_K$  in Example 5.7, we obtain a simple  $\mathbf{H}_{K,v}$ -module denoted by  $E_v$ . Thus, we have the correspondences:

$$E \in \text{Irr}_{\mathbb{Q}}(W) \quad \leftrightarrow \quad E_{\spadesuit} \in \text{Irr}(\mathbf{J}_{\mathbb{Q}}) \quad \leftrightarrow \quad E_v \in \text{Irr}(\mathbf{H}_{K,v}).$$

We can also express this in terms of characters. As in the previous section, denote by  $\mathbf{H}_{K,v}^\wedge$  the set of irreducible characters of  $\mathbf{H}_{K,v}$ . The set  $\mathbf{J}_{\mathbb{Q}}^\wedge$  is defined similarly. Then we also have bijective correspondences:

$$\chi \in W^\wedge \quad \leftrightarrow \quad \chi_{\spadesuit} \in \mathbf{J}_{\mathbb{Q}}^\wedge \quad \leftrightarrow \quad \chi_v \in \mathbf{H}_{K,v}^\wedge.$$

(If  $\chi$  is afforded by  $E \in \text{Irr}_{\mathbb{Q}}(W)$ , then  $\chi_{\spadesuit}$  is the character afforded by  $E_{\spadesuit}$  and  $\chi_v$  is the character afforded by  $E_v$ .) We have  $\chi(w) \in \mathbb{Z}$  for all  $w \in W$ . Similarly, we have  $\chi_{\spadesuit}(t_w) \in \mathbb{Z}$  and  $\chi_v(\tilde{T}_w) \in A$  for all  $w \in W$ . This follows by a general argument, since  $\mathbb{Z}$  is integrally closed in  $\mathbb{Q}$  and  $A$  is integrally closed in  $K$ ; see [47, 7.3.9]. The discussion in Example 5.7 easily implies that

$$\chi(w) = \chi_v(T_w)|_{v=1} \quad \text{for all } w \in W;$$

see [73, 20.3]. Thus, we have recovered Theorem 4.3 by a general argument (assuming that P2–P8 and P15' hold).

Now the general philosophy will be that, although we don't really know the algebra  $\mathbf{J}$  explicitly, it will serve as a theoretical tool for various module-theoretic constructions. We will see the full power of this in the following section, where we consider non-semisimple specialisations. Here, on the “generic level”, we now use this idea to give a new interpretation to the

invariants  $\alpha_\chi$  and  $f_\chi$  defined by the formula in Theorem 4.4. This is done as follows. We have a direct sum decomposition

$$\mathbf{J} = \bigoplus_{a \geq 0} \mathbf{J}^a \quad \text{where} \quad \mathbf{J}^a := \langle t_w \mid w \in W \text{ such that } \mathbf{a}(w) = a \rangle_{\mathbb{Z}}.$$

By P7, each  $\mathbf{J}^a$  is a two-sided ideal in  $\mathbf{J}$ . In fact, one easily checks that

$$t_a := \sum_{\substack{d \in W \\ \mathbf{a}(d) = a}} n_d t_d \in \mathbf{J}^a$$

is a central idempotent in  $\mathbf{J}$ ; furthermore, we have  $1_J = \sum_{a \geq 0} t_a$  and  $t_a t_{a'} = 0$  for  $a \neq a'$ . Hence, for any simple  $M \in \text{Irr}(\mathbf{J}_{\mathbb{Q}})$ , there exists a unique  $a \geq 0$  such that  $t_a.M = M$  and  $t_{a'}.M = \{0\}$  for  $a' \neq a$ . Note that we have

$$\forall z \in W : \quad t_z.M \neq \{0\} \quad \Rightarrow \quad \mathbf{a}(z) = a.$$

**Proposition 5.8** (Lusztig). *Let  $E \in \text{Irr}_{\mathbb{Q}}(W)$  and consider the corresponding  $\mathbf{J}_{\mathbb{Q}}$ -module  $E_{\spadesuit}$ . Then we have  $\alpha_E = a$  where  $a \geq 0$  is uniquely determined by the condition that  $t_a.E_{\spadesuit} = E_{\spadesuit}$  (and  $t_{a'}.E_{\spadesuit} = \{0\}$  for  $a' \neq a$ ).*

Let us sketch the main arguments of the proof, following Lusztig [73, Chap. 20]. Let  $\chi$  be the character afforded by  $E \in \text{Irr}_{\mathbb{Q}}(W)$ . Let  $a \geq 0$  be such that  $t_a.E_{\spadesuit} = E_{\spadesuit}$ , as above. First note that, by definition, we have

$$\chi_v(c_w^\dagger) = \chi_{\spadesuit}(\phi_{K,v}(c_w^\dagger)) = \sum_{\substack{z \in W, d \in \mathcal{D} \\ \mathbf{a}(z) = \mathbf{a}(d)}} h_{w,d,z} \hat{n}_z \chi_{\spadesuit}(t_z),$$

where the sum need only be extended over all  $z \in W$  such that  $\mathbf{a}(z) = a$ . Using the properties P2–P7, one shows that

$$v^{\mathbf{a}(w)} \chi_v(c_w^\dagger) \in \mathbb{Z}[v] \quad \text{and} \quad v^{\mathbf{a}(w)} \chi_v(c_w^\dagger) \equiv \chi_{\spadesuit}(t_w) \pmod{v\mathbb{Z}[v]}$$

for all  $w \in W$ . Now we claim that we have

$$(*) \quad a = \max\{\mathbf{a}(w) \mid w \in W \text{ and } \chi_v(c_w^\dagger) \neq 0\}.$$

First note that  $\chi_{\spadesuit}(t_w) \neq 0$  for some  $w \in W$  and, hence,  $\chi_v(c_w^\dagger) \neq 0$ . Since  $\mathbf{a}(w) = a$ , we have the inequality “ $\leq$ ”. On the other hand, assume that  $\chi_v(c_w^\dagger) \neq 0$ . Then there exist some  $z \in W$ ,  $d \in \mathcal{D}$  such that  $h_{w,d,z} \neq 0$  and  $\mathbf{a}(z) = a$ . Now P4 shows that  $\mathbf{a}(w) \leq \mathbf{a}(z)$ . Thus,  $(*)$  holds.

Consequently, we have

$$v^a \chi_v(c_w^\dagger) \in \mathbb{Z}[v] \quad \text{and} \quad v^a \chi_v(c_w^\dagger) \equiv \chi_{\spadesuit}(t_w) \pmod{v\mathbb{Z}[v]}$$

for all  $w \in W$ . Now recall that  $c_w \equiv \tilde{T}_w \pmod{\mathbf{H}_{<0}}$ . Since  $\bar{c}_w = c_w$ , we have  $c_w^\dagger = j(c_w)$  and so  $c_w^\dagger \equiv (-1)^{l(w)} \tilde{T}_w \pmod{\mathbf{H}_{>0}}$ . These relations imply that we can write

$$\tilde{T}_w = (-1)^{l(w)} c_w^\dagger + v\mathbb{Z}[v]\text{-combination of various } c_y^\dagger \ (y \in W).$$

Hence we also have

$$v^a \chi_v(\tilde{T}_w) \in \mathbb{Z}[v] \quad \text{and} \quad v^a \chi_v(\tilde{T}_w) \equiv (-1)^{l(w)} \chi_{\spadesuit}(t_w) \pmod{v\mathbb{Z}[v]}$$

for all  $w \in W$ . Inserting the above congruence conditions into the orthogonality relations for the irreducible characters of  $\mathbf{H}_{k,v}$  from the previous section, we deduce that

$$v^{2a} \chi(1) \mathbf{c}_\chi \equiv \left( \sum_{w \in W} \chi_\spadesuit(t_w) \chi_\spadesuit(t_{w^{-1}}) \right) \bmod v\mathbb{Z}[v].$$

Finally, by [73, 20.1],  $\mathbf{J}$  also is a symmetric algebra, with trace function  $\mu: \mathbf{J} \rightarrow \mathbb{Z}$  given by

$$\mu(t_z) = \begin{cases} n_z & \text{if } z \in \mathcal{D}, \\ 0 & \text{otherwise.} \end{cases}$$

We have  $\mu(t_x t_y) = 1$  if  $x = y^{-1}$  and  $\mu(t_x t_y) = 0$  otherwise. (To check this, use P2, P3, P5, P6). Hence, we can write

$$\mu_{\mathbb{Q}} = \sum_{\psi \in W^\wedge} d_\psi^{-1} \psi_\spadesuit \quad \text{where} \quad 0 \neq d_\psi \in \mathbb{Z}.$$

By the orthogonality relations for the irreducible characters of  $\mathbf{J}_{\mathbb{Q}}$ , we have

$$\sum_{w \in W} \chi_\spadesuit(t_w) \chi_\spadesuit(t_{w^{-1}}) = \chi(1) d_\chi.$$

A comparison with the above formula shows that

$$\mathbf{c}_\chi = d_\chi v^{-2a} + \text{combination of higher powers of } v.$$

Hence, we must have  $a = \alpha_\chi$  and  $d_\chi = f_\chi$ , as desired. Thus, both  $\alpha_\chi$  and  $f_\chi$  can be interpreted in terms of the algebra  $\mathbf{J}$ .  $\square$

*Remark 5.9.* One can actually check that all the arguments in this section go through in the case where we consider a weight function  $L: W \rightarrow \mathbb{N}$  and allow the possibility that  $L(s) = 0$  for some  $s \in S$ . (In fact, Lusztig [73, 5.2] proves Theorem 5.1 in this more general set-up. Some care is needed in the definition of  $\mathcal{D}$  since it may happen that  $\tau(c_z) = 0$  for  $z \in W$ . Everything works out well by setting  $\Delta(z) = \infty$  if  $\tau(c_z) = 0$ .)

## 6. CANONICAL BASIC SETS AND LUSZTIG'S RING $J$

We keep the setting of the previous section. Now let  $\theta: A \rightarrow k$  be a ring homomorphism into a field  $k$  whose characteristic is either zero or a prime which is  $L$ -good; see Definition 4.5. As before, let

$$\mathbf{H}_{k,\xi} = k \otimes_A \mathbf{H} \quad \text{where} \quad \xi = \theta(v^2).$$

Our aim is to establish a general existence result for “canonical basic sets”  $\mathcal{B}_{k,\xi}$  as in Definition 4.13. In doing so, we will also have to deal with the question of splitting fields for the algebra  $\mathbf{H}_{k,\xi}$ . By Remark 4.14, the critical case that we have to study is the case where  $\xi$  is a root of unity in  $k$ .

We assume throughout that P2–P8 and P15’ hold for  $W, L$ ; see (5.2). The starting point are the following two results.

**Proposition 6.1.** *Recall our assumptions on the characteristic of  $k$ . Then  $\mathbf{J}_k$  is semisimple and we have a unique bijection*

$$\mathrm{Irr}(\mathbf{J}_{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{J}_k), \quad M \mapsto M^k$$

*such that  $\dim M = \dim M^k$  and  $\mathrm{trace}(t_w, M^k) = \theta(\mathrm{trace}(t_w, M))$  for all  $w \in W$ .*

*Proof.* (See [29, 2.5].) Our assumption on the characteristic implies that  $\theta(f_E) \neq 0$  for all  $E \in \mathrm{Irr}_{\mathbb{Q}}(W)$ . The relevance of the numbers  $f_E$  here is the fact that they are the coefficients in the expansion of the trace function  $\mu_{\mathbb{Q}}$  in terms of irreducible characters (see the discussion at the end of the previous section). Hence, by the same argument as in Theorem 4.11, we conclude that  $\mathbf{J}_k$  is split semisimple. The fact that we have a unique bijection between the simple modules with the above properties is a consequence of *Tits' Deformation Theorem*; see [47, 7.4.6].  $\square$

**Corollary 6.2.** *We have a “canonical” bijection*

$$\mathrm{Irr}_{\mathbb{Q}}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{J}_k), \quad \text{denoted } E \mapsto E_{\spadesuit}^k.$$

*For  $E \in \mathrm{Irr}_{\mathbb{Q}}(W)$ , we have  $\dim E = \dim E_{\spadesuit}^k$ ; furthermore,  $\alpha_E = \mathbf{a}(z)$  for any  $z \in W$  such that  $t_z.E_{\spadesuit}^k \neq \{0\}$ .*

*Proof.* We have a bijection  $\mathrm{Irr}_{\mathbb{Q}}(W) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{J}_{\mathbb{Q}})$  induced by the algebra isomorphism in Example 5.7. The correspondence  $E \mapsto E_{\spadesuit}^k$  is obtained by composing that bijection with the one in Proposition 6.1. Now consider the statement concerning the  $a$ -invariants. As in the previous section,

$$t_a := \sum_{\substack{d \in W \\ \mathbf{a}(d)=a}} n_d t_d \in \mathbf{J}_k^a \quad (a = 0, 1, 2, \dots)$$

are central idempotents; furthermore, we have  $1_J = \sum_{a \geq 0} t_a$  and  $t_a t_{a'} = 0$  for  $a \neq a'$ . Hence, there exists a unique  $a_0$  such that  $t_{a_0}.E_{\spadesuit}^k = E_{\spadesuit}^k$ . In particular, we have  $\mathbf{a}(z) = a_0$  for all  $z \in W$  such that  $t_z.E_{\spadesuit}^k \neq \{0\}$ . In order to show that  $a_0 = \alpha_E$ , it will now be sufficient to find some element  $z_0 \in W$  such that  $\mathbf{a}(z_0) = \alpha_E$  and  $t_{z_0}.E_{\spadesuit}^k \neq \{0\}$ . This is seen as follows. Since  $\mathbf{J}_k$  is split semisimple, the character of a simple module cannot be identically zero. So there exists some  $z_0 \in W$  such that  $\mathrm{trace}(t_{z_0}, E_{\spadesuit}^k) \neq 0$ . If we had  $\mathbf{a}(z_0) \neq \alpha_E$ , then  $t_{z_0}.E_{\spadesuit}^k = \{0\}$  and so Proposition 6.1 would yield

$$\mathrm{trace}(t_{z_0}, E_{\spadesuit}^k) = \theta(\mathrm{trace}(t_{z_0}, E_{\spadesuit})) = 0,$$

a contradiction. Hence, we must have  $\mathbf{a}(z_0) = \alpha_E$ .  $\square$

In order to obtain a canonical basic set for  $\mathbf{H}_{k,\xi}$ , we need to construct a map  $\mathrm{Irr}(\mathbf{H}_{k,\xi}) \rightarrow \mathrm{Irr}_{\mathbb{Q}}(W)$ . By Corollary 6.2, this is equivalent to constructing a map  $\mathrm{Irr}(\mathbf{H}_{k,\xi}) \rightarrow \mathrm{Irr}(\mathbf{J}_k)$ .

Following Lusztig (see the proof of [70, Lemma 1.9]), we attach an integer  $\alpha_M$  to any  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$  by the requirement that

$$\begin{aligned} c_w^\dagger.M &= \{0\} \quad \text{for all } w \in W \text{ with } \mathbf{a}(w) > \alpha_M, \\ c_w^\dagger.M &\neq \{0\} \quad \text{for some } w \in W \text{ with } \mathbf{a}(w) = \alpha_M. \end{aligned}$$

Now consider the  $(\mathbf{J}_k, \mathbf{H}_{k,\xi})$ -bimodule  $\mathbf{H}_{k,\xi}^a$  where  $a := \alpha_M$ . Then we obtain a (left)  $\mathbf{J}_k$ -module

$$\tilde{M} := \mathbf{H}_{k,\xi}^a \otimes_{\mathbf{H}_{k,\xi}} M.$$

**Lemma 6.3.** *Let  $E \in \text{Irr}_{\mathbb{Q}}(W)$  and assume that  $E_{\spadesuit}^k \in \text{Irr}(\mathbf{J}_k)$  is a composition factor of  $\tilde{M}$ . Then  $\alpha_E = \alpha_M$ .*

*Proof.* (See the proof of [70, Cor. 3.6].) Let  $z \in W$  be such that  $\mathbf{a}(z) = \alpha_E$  and  $t_z.E_{\spadesuit}^k \neq 0$ . Then we also have  $t_z.\tilde{M} \neq \{0\}$  and so  $t_z \star [c_w^\dagger] \neq 0$  for some  $w \in W$  such that  $\mathbf{a}(w) = a$ . The defining formula for “ $\star$ ” and P8 now imply that  $\alpha_E = \mathbf{a}(z) = \mathbf{a}(w) = \alpha_M$ .  $\square$

Now there is a well-defined  $k$ -linear map  $\pi: \tilde{M} \rightarrow M$  such that

$$\pi([f] \otimes m) = f.m \quad \text{for any } f \in \mathbf{H}_{k,\xi}^{\geq a} \text{ and } m \in M.$$

Note that, if  $f', f \in \mathbf{H}_{k,\xi}^{\geq a}$  are such that  $[f] = [f']$  in  $\mathbf{H}_{k,\xi}^a$ , then  $f - f' \in \mathbf{H}_{k,\xi}^{\geq a+1}$  acts as zero on  $M$  by the definition of  $a = \alpha_M$ .

Now, we may also consider  $\tilde{M}$  as an  $\mathbf{H}_{k,\xi}$ -module, using the map  $\mathbf{J}_k\text{-mod} \rightarrow \mathbf{H}_{k,\xi}\text{-mod}$  defined as follows. If  $V$  is any  $\mathbf{J}_k$ -module, we can regard  $V$  as an  $\mathbf{H}_{k,\xi}$  by composing the action of  $\mathbf{J}_k$  on  $V$  with the algebra homomorphism  $\phi_{k,\xi}: \mathbf{H}_{k,\xi} \rightarrow \mathbf{J}_k$ . We denote that  $\mathbf{H}_{k,\xi}$ -module by  $*V$ . Note that, since  $\mathbf{H}_{k,\xi}$  is not necessarily semisimple,  $\phi_{k,\xi}$  may not be an isomorphism and, consequently,  $*V$  may not be simple.

Using this notation, let us consider the  $\mathbf{H}_{k,\xi}$ -module  $*\tilde{M}$ . Note that, by the compatibility in Theorem 5.4, the left  $\mathbf{H}_{k,\xi}$ -module structure on  $*\tilde{M}$  is just the one coming from the natural left action of  $\mathbf{H}_{k,\xi}$  on  $\mathbf{H}_{k,\xi}^a$ .

**Lemma 6.4.**  *$\pi: *\tilde{M} \rightarrow M$  is a surjective homomorphism of  $\mathbf{H}_{k,\xi}$ -modules. If  $M' \in \text{Irr}(\mathbf{H}_{k,\xi})$  is a composition factor of  $\ker(\pi)$ , then  $\alpha_{M'} < \alpha_M$ . In particular,  $M$  occurs with multiplicity 1 as a composition factor of  $*\tilde{M}$ .*

*Proof.* (See [70, Lemma 1.9].) Let  $h \in \mathbf{H}_{k,\xi}$ ,  $f \in \mathbf{H}_{k,\xi}^{\geq a}$  and  $m \in M$ . Then  $hf \in \mathbf{H}_{k,\xi}^{\geq a}$  by P4 and so

$$h.\pi([f] \otimes m) = h.(f.m) = (hf).m = \pi([hf] \otimes m) = \pi(h.([f] \otimes m)).$$

Thus,  $\pi$  is a homomorphism of  $\mathbf{H}_{k,\xi}$ -modules. By the definition of  $a = \alpha_M$ , there exists some  $m \in M$  and some  $w \in W$  such that  $\mathbf{a}(w) = a$  and  $\pi([c_w^\dagger] \otimes m) = c_w^\dagger.m \neq 0$ . Thus,  $\pi \neq 0$ . Since  $M$  is simple, we conclude that  $\pi$  is surjective. As far as the composition factors of  $\ker(\pi)$  are concerned, it will be sufficient to show that  $c_w^\dagger.\tilde{m} = 0$  for any  $\tilde{m} \in \ker(\pi)$  and  $w \in W$



such that  $\mathbf{a}(w) \geq a$ . So let  $\tilde{m} := \sum_i [f_i] \otimes m_i \in \ker(\pi)$  where  $f_i \in \mathbf{H}_{k,\xi}^{\geq a}$  and  $m_i \in M$ . Then

$$\begin{aligned} c_w^\dagger \cdot \tilde{m} &= \left( \sum_i [f_i] \otimes m_i \right) = \sum_i \left( c_w^\dagger \cdot [f_i] \right) \otimes m_i \\ &= \sum_i [c_w^\dagger f_i] \otimes m_i = \sum_i [c_w^\dagger] \cdot f_i \otimes m_i \quad \text{since } \mathbf{a}(w) \geq a \\ &= c_w^\dagger \cdot \left( \sum_i f_i \cdot m_i \right) = c_w^\dagger \cdot \pi(\tilde{m}) = 0, \end{aligned}$$

as desired.  $\square$

**Corollary 6.5.** *Let  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ . Then there is a unique  $E = E(M) \in \text{Irr}_{\mathbb{Q}}(W)$  such that*

- (a)  $E_{\spadesuit}^k$  is a composition factor of  $\tilde{M}$  and
- (b)  $M$  is a composition factor of  ${}^*E_{\spadesuit}^k$ .

*The multiplicity of  $M$  as a composition factor of  ${}^*E_{\spadesuit}^k$  is 1 and  $\alpha_M = \alpha_E$ . For  $M, M' \in \text{Irr}_{\mathbb{Q}}(W)$ , we have  $E(M) \cong E(M')$  if and only if  $M \cong M'$ .*

*Proof.* Consider a composition series of  $\tilde{M}$  as a  $\mathbf{J}_k$ -module. Now, composing the action of  $\mathbf{J}_k$  with the homomorphism  $\phi_{k,\xi}: \mathbf{H}_{k,\xi} \rightarrow \mathbf{J}_k$ , the composition series becomes a filtration of  ${}^*\tilde{M}$  as an  $\mathbf{H}_{k,\xi}$ -module, where the factors are not necessarily simple. Note, however, that all of these factors are of the form  ${}^*E_{\spadesuit}^k$ , where  $E \in \text{Irr}_{\mathbb{Q}}(W)$  and  $E_{\spadesuit}^k$  is a composition factor of  $\tilde{M}$ . By Lemma 6.4, there is a unique such factor  ${}^*E_{\spadesuit}^k$  which has  $M$  as a composition factor. This defines  $E = E(M)$ .

It remains to show that  $E(M) \not\cong E(M')$  if  $M \not\cong M'$ . Now, by Lemma 6.3, we have  $\alpha_{E(M)} = \alpha_M$  and  $\alpha_{E(M')} = \alpha_{M'}$ . Hence, if  $\alpha_M \neq \alpha_{M'}$ , then we certainly have  $E(M) \not\cong E(M')$ . Now suppose that  $\alpha_M = \alpha_{M'}$ . Since  $E(M)_{\spadesuit}^k$  is a composition factor of  $\tilde{M}$ , Lemma 6.4 shows that  $M$  is the unique composition factor of  $E(M)_{\spadesuit}^k$  which has  $a$ -invariant  $\alpha_M$ . Thus,  $M'$  cannot be a composition factor of  $E(M)_{\spadesuit}^k$  and so  $E(M)_{\spadesuit}^k \not\cong E(M')_{\spadesuit}^k$ . By Corollary 6.2, this shows that  $E(M) \not\cong E(M')$ , as required.  $\square$

The above result defines an injective map

$$\text{Irr}(\mathbf{H}_{k,\xi}) \hookrightarrow \text{Irr}_{\mathbb{Q}}(W), \quad M \mapsto E(M).$$

We have  $\alpha_M = \alpha_{E(M)}$  and  $[E(M) : M] = 1$  for all  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ . Let

$$\mathcal{B}_{k,\xi} := \{E \in \text{Irr}_{\mathbb{Q}}(W) \mid E = E(M) \text{ for some } M \in \text{Irr}(\mathbf{H}_{k,\xi})\}.$$

Now we can state the main result of this section.

**Theorem 6.6** (Geck [29], [32] and Geck–Rouquier [48]). *Recall that the characteristic of  $k$  is assumed to be zero or a prime which is  $L$ -good. Then  $\mathbf{H}_{k,\xi}$  is split and  $\mathcal{B}_{k,\xi}$  is a canonical basic set in the sense of Definition 4.13.*

First let us sketch the proof that  $\mathbf{H}_{k,\xi}$  is split. Let  $k_0 \subseteq k$  be the field of fractions of the image of  $\theta$ . In order to prove that  $\mathbf{H}_{k,\xi}$  is split, it is certainly enough to show that  $\mathbf{H}_{k_0,\xi}$  is split. By Remark 4.14, we may assume that  $\xi$  is a root of unity in  $k_0$ . Hence,  $k_0$  is a finite extension of the prime field of  $k_0$ . Consequently,  $k_0$  is either a finite field or a finite extension of  $\mathbb{Q}$ ; in particular,  $k_0$  is perfect. Now Lemma 6.3, Lemma 6.4 and Corollary 6.5 also hold with  $k$  replaced by  $k_0$ . Then we can argue as in [32, Theorem 3.5] to prove that  $\mathbf{H}_{k_0,\xi}$  is split, where standard results on “Schur indices” and the “multiplicity 1” statement in Corollary 6.5 play a crucial role.

In order to show that  $\mathcal{B}_{k,\xi}$  is a canonical basic set, we can now assume that  $k = k_0$ .

We will need some standard results on projective indecomposable  $\mathbf{H}_{k,\xi}$ -modules (PIM’s for short). Let  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ . Then there is a PIM  $P = P(M)$  (unique up to isomorphism) such that  $M$  is the unique simple quotient of  $P$ . We can assume that  $P = \mathbf{H}_{k,\xi}e$  where  $e \in \mathbf{H}_{k,\xi}$  is a primitive idempotent. Let us consider the filtration of  $\mathbf{H}_{k,\xi}$  defined by the  $\mathbf{a}$ -function:

$$\{0\} = \mathbf{H}_{k,\xi}^{\geq N+1} \subseteq \mathbf{H}_{k,\xi}^{\geq N} \subseteq \mathbf{H}_{k,\xi}^{\geq N-1} \subseteq \cdots \subseteq \mathbf{H}_{k,\xi}^{\geq 0} = \mathbf{H}_{k,\xi}$$

where  $N = \max\{\mathbf{a}(w) \mid w \in W\}$ . Multiplying on the right by  $e$  and setting  $P^{\geq i} := \mathbf{H}_{k,\xi}^{\geq i}e$ , we obtain a filtration

$$\{0\} = P^{\geq N+1} \subseteq P^{\geq N} \subseteq P^{\geq N-1} \subseteq \cdots \subseteq P^{\geq 0} = P.$$

Now fix  $i$  and consider the canonical exact sequence

$$\{0\} \rightarrow \mathbf{H}_{k,\xi}^{\geq i+1} \rightarrow \mathbf{H}_{k,\xi}^{\geq i} \rightarrow \mathbf{H}_{k,\xi}^i \rightarrow \{0\}.$$

Multiplying with  $e$  yields a sequence

$$\{0\} \rightarrow \mathbf{H}_{k,\xi}^{\geq i+1}e \rightarrow \mathbf{H}_{k,\xi}^{\geq i}e \rightarrow \mathbf{H}_{k,\xi}^ie \rightarrow \{0\}.$$

which is also exact. (To see this, also work with the idempotent  $1 - e$ .) It follows that we have a canonical isomorphism of  $\mathbf{H}_{k,\xi}$ -modules

$$P^i \cong \mathbf{H}_{k,\xi}^ie \quad \text{for } i = 0, 1, 2, \dots$$

Now consider the  $(\mathbf{J}_k, \mathbf{H}_{k,\xi})$ -bimodule structure of  $\mathbf{H}_{k,\xi}^i$ ; see Theorem 5.4. Since  $\mathbf{H}_{k,\xi}^ie$  is obtained by right multiplication with some element of  $\mathbf{H}_{k,\xi}$ , it is clear that  $\mathbf{H}_{k,\xi}^ie$  is a  $\mathbf{J}_k$ -submodule of  $\mathbf{H}_{k,\xi}^i$ . Hence, we can write

$$P^i \cong \mathbf{H}_{k,\xi}^ie = V_1 \oplus \cdots \oplus V_r \quad \text{where } V_1, \dots, V_r \in \text{Irr}(\mathbf{J}_k).$$

Note that, since  $P^i$  is isomorphic to a submodule of  $\mathbf{H}_{k,\xi}^i$ , this implies that  $\alpha_{V_1} = \cdots = \alpha_{V_r} = i$ . This leads us to the following result.

**Proposition 6.7** (Brauer reciprocity). *Let  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$  and  $P = P(M)$  be a corresponding PIM. For any  $E \in \text{Irr}(\mathbf{J}_k)$  such that  $\alpha_E = i$ , we have*

$$\begin{aligned} [E : M] &= \text{multiplicity of } M \text{ as a composition factor of } {}^*E_{\clubsuit}^k \\ &= \text{multiplicity of } E_{\clubsuit}^k \text{ as a direct summand in } P^i, \end{aligned}$$

where  $P^i$  is considered as a left  $\mathbf{J}_k$ -module as explained above.

The proof of the first equality uses an abstract characterization of the multiplicities  $[E : M]$  in terms of decomposition maps between the appropriate Grothendieck groups; see [30, §2] and [29, 2.5]. Then the second statement can be reduced to the classical Brauer reciprocity in the modular representation theory of finite groups and associative algebras; see [48] for the details.

Now we can complete the proof of Theorem 6.6. Let  $E \in \text{Irr}_{\mathbb{Q}}(W)$  and  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ . First we show:

$$(*) \quad [E : M] \neq 0 \quad \Rightarrow \quad \alpha_M \leq \alpha_E.$$

This is seen as follows. Let  $w \in W$  be such that  $\mathbf{a}(w) = \alpha_M$  and  $c_w^\dagger.M \neq \{0\}$ . Then  $c_w^\dagger$  also acts non-zero on  $E_{\clubsuit}^k$ . By the definition of that action, this means that  $\phi_{k,\xi}(c_w^\dagger).E_{\clubsuit}^k \neq \{0\}$ . So there exist some  $z \in W$  and  $d \in \mathcal{D}$  such that  $h_{w,d,z} \neq 0$  and  $t_z.E_{\clubsuit}^k \neq \{0\}$ . Thus, we necessarily have  $\alpha_E = \mathbf{a}(z)$  by Corollary 6.2. Now P4 implies that  $\mathbf{a}(w) \leq \mathbf{a}(z)$ , as required.

Next we show that

$$\check{\alpha}_M = \alpha_M \quad \text{for any } M \in \text{Irr}(\mathbf{H}_{k,\xi}).$$

Indeed, let  $E \in \text{Irr}_{\mathbb{Q}}(W)$  be such that  $[E : M] \neq 0$  and  $\check{\alpha}_M = \alpha_E$ . Then  $(*)$  shows that  $\alpha_M \leq \check{\alpha}_M$ . On the other hand,  $[E(M) : M] = 1$  by Corollary 6.5. Hence  $\check{\alpha}_M \leq \alpha_{E(M)} = \alpha_M$ , where the last equality holds by Lemma 6.3.

Finally, we must show that, for any  $E \in \text{Irr}_{\mathbb{Q}}(W)$  and  $M \in \text{Irr}(\mathbf{H}_{k,\xi})$ , we have the implication:

$$[E : M] \neq 0 \quad \text{and} \quad \alpha_M = \alpha_E \quad \Rightarrow \quad E = E(M).$$

To prove this, we consider a PIM  $P = P(M)$ . Let  $a := \min\{i \geq 0 \mid P^i \neq \{0\}\}$ . Then  $P = P^a$  and there is a natural surjective homomorphism  $P \rightarrow P^a$  of  $\mathbf{H}_{k,\xi}$ -modules. As before, considering  $P^a$  as a left  $\mathbf{J}_k$ -module, we can write

$$P^a \cong \mathbf{H}_{k,\xi}^a e = V_1 \oplus \cdots \oplus V_r \quad \text{where } V_1, \dots, V_r \in \text{Irr}(\mathbf{J}_k).$$

The compatibility in Theorem 5.4 now implies that we have an isomorphism

$$P^a \cong {}^*V_1 \oplus \cdots \oplus {}^*V_r \quad (\text{as left } \mathbf{H}_{k,\xi}\text{-modules}).$$

Since  $M$  is the unique simple quotient of  $P$ , we deduce that  $M$  also is the unique simple quotient of  $P^a$ . In particular,  $P^a$  is indecomposable. Hence we must have  $r = 1$ . Let us write  $V_1 = E_{1,\clubsuit}^k$  where  $E_1 \in \text{Irr}_{\mathbb{Q}}(W)$ . By Proposition 6.7, we have  $[E_1 : M] \neq 0$  and so  $\alpha_M \leq \alpha_{E_1} = a$ , using  $(*)$ .

Now consider  $E$  such that  $[E : M] \neq 0$  and  $\alpha_M = \alpha_E$ . By Proposition 6.7 and the definition of  $a$ , we have  $\alpha_E \geq a$ . Furthermore, if  $\alpha_E = a$ , then we necessarily have  $E \cong E_1$ . Now, since  $\alpha_M = \alpha_E$ , we cannot have  $\alpha_E > a$ . Hence  $a = \alpha_E$  and so  $E \cong E_1$ .

Finally, we do have  $[E(M) : M] \neq 0$  and  $\alpha_M = \alpha_{E(M)}$ . Hence, we can apply the previous argument to  $E(M)$  and this yields  $E(M) \cong E_1 \cong E$ , as required. This completes the proof of Theorem 6.6.

**Example 6.8.** Assume that  $L$  is a positive multiple of the length function. Then, as mentioned in (5.2), the properties P1–P15 are known to hold. Hence, the above results guarantee the existence of a canonical basic set for  $\mathbf{H}_{k,\xi}$  if the characteristic of  $k$  is zero or a good prime for  $W$ . Explicit tables for these canonical basic sets for the exceptional types  $G_2, F_4, E_6, E_7, E_8$  are given by Jacon [55, §3.3], using the known information on decomposition numbers (see [46], [27], [28] and Müller [79]).

**Example 6.9.** Let  $W_n$  be a Coxeter group of type  $B_n$ , as in Example 4.9. Recall that a weight function  $L$  on  $W_n$  is specified by two integers  $a, b \geq 0$ . Let us denote the corresponding specialized algebra simply by  $H_n$ .

Now, as in the case of the symmetric group (see Example 4.12), there is a theory of “Specht modules” for  $H_n$ ; see Dipper–James–Murphy [23]. Thus, for any bipartition  $\lambda \vdash n$ , we have a corresponding Specht module  $S_k^\lambda \in H_n$ -mod. (Actually, we shall denote by  $S_k^\lambda$  the module that is labelled by  $(\lambda_{(2)}^*, \lambda_{(1)}^*)$  in [23], where the star denotes the conjugate partition.) As before, this module has the property that

$$[E^\lambda : M] = \text{multiplicity of } M \text{ as a composition factor of } S_k^\lambda,$$

for any  $M \in \text{Irr}(H_n)$ . Furthermore, there is an  $H_n$ -equivariant symmetric bilinear form on  $S_k^\lambda$  and, taking quotients by the radical, we obtain modules

$$D^\lambda := S_k^\lambda / \text{rad}(S_k^\lambda) \in H_n\text{-mod}.$$

Let  $\Lambda_{2,n}^\circ := \{\lambda \vdash n \mid D^\lambda \neq \{0\}\}$ . Then, by [23, §6], we have

$$\text{Irr}(H_n) = \{D^\lambda \mid \lambda \in \Lambda_{2,n}^\circ\}.$$

Furthermore, the decomposition numbers  $[S^\lambda : D^\mu]$  satisfy conditions similar to those in Example 4.12, where we have to consider the dominance order on bipartitions.

Is it possible to interpret  $\Lambda_{2,n}^\circ$  in terms of canonical basic sets? The answer is “yes”, but we need to choose the parameters  $a, b$  appropriately. So let us assume that  $b > (n-1)a > 0$ , that is, we are in the “asymptotic case” defined in Example 4.9. All primes are  $L$ -good in this case. Furthermore, as remarked in (5.2), the properties P2–P8 and P15’ hold. Hence, Theorem 6.6 shows that we have a canonical basic set  $\mathcal{B}_{k,\xi}$ . Using a compatibility of the dominance order on bipartitions with the invariants  $\alpha_E$  for this case (see [43, Cor. 5.5]), one can use exactly the same arguments as in Example 4.12

to show that

$$\mathcal{B}_{k,\xi} = \{E^\lambda \mid \lambda \in \Lambda_{2,n}^\circ\} \quad (\text{assuming that } b > (n-1)a > 0).$$

However, contrary to the situation in the symmetric group, a combinatorial description of  $\Lambda_{2,n}^\circ$  is much harder to obtain. Dipper, James and Murphy already dealt with the following cases. As in Example 4.12, let us set

$$e := \min\{i \geq 2 \mid 1 + \xi^a + \xi^{2a} + \cdots + \xi^{(i-1)a} = 0\}.$$

(If no such  $i$  exists, we set  $e = \infty$ .) Following Dipper–James [21, 4.4], let

$$f_n(a, b) := \prod_{i=-(n-1)}^{n-1} (\xi^b + \xi^{ai}).$$

Now there is a distinction between the cases where  $f_n(a, b)$  is zero or not.

- **The case**  $f_n(a, b) \neq 0$ . Then we have

$$\Lambda_{2,n}^\circ = \{\lambda \vdash n \mid \lambda_{(1)} \text{ and } \lambda_{(2)} \text{ are } e\text{-regular}\}.$$

Indeed, by [23, Theorem 6.9], we have the inclusion “ $\subseteq$ ”. On the other hand, Dipper–James [21, 4.17 and 5.3] showed that there is a Morita equivalence

$$H_n\text{-mod} \cong \left( \bigoplus_{r=0}^n H_k(\mathfrak{S}_r, \xi^a) \otimes_k H_k(\mathfrak{S}_{n-r}, \xi^a) \right)\text{-mod};$$

In particular, we have a bijection

$$\text{Irr}(H_n) \xrightarrow{\sim} \prod_{r=0}^n \left( \text{Irr}(H_k(\mathfrak{S}_r, \xi^a)) \times \text{Irr}(H_k(\mathfrak{S}_{n-r}, \xi^a)) \right).$$

Consequently, there are as many simple  $H_n$ -modules as there are bipartitions  $\lambda$  such that  $\lambda_{(1)}$  and  $\lambda_{(2)}$  are  $e$ -regular. This yields the inclusion “ $\supseteq$ ”.

In [45], we show that, for *any* values of  $a, b \geq 0$  such that  $f_n(a, b) \neq 0$ , the set  $\Lambda_{2,n}^\circ$  is a canonical basic set in the sense of Definition 4.13, provided the characteristic of  $k$  is not 2.

- **The case**  $\xi^a = 1$ . If  $\xi^b \neq -1$ , then  $f_n(a, b) \neq 0$  and we can apply the previous case. So let us now assume that  $\xi^b = -1$ . The special feature of this case is that, by [21, Remark 5.4], the simple  $H_n$ -modules are obtained by extending (in a unique way) the simple modules of the parabolic subalgebra  $k\mathfrak{S}_n = \langle T_{s_1}, \dots, T_{s_{n-1}} \rangle_k$  to  $H_n$ . By [23, Theorem 7.3], we have

$$\Lambda_{2,n}^\circ = \{\lambda \vdash n \mid \lambda_{(1)} \text{ is } e\text{-regular and } \lambda_{(2)} = \emptyset\}.$$

The general case where  $f_n(a, b) = 0$  will be discussed in Section 8.

The above discussion shows that the set  $\Lambda_{2,n}^\circ$  arising from the theory of Specht modules can be interpreted as a *canonical basic set* assuming that  $b$  is large with respect to  $a$ . It would certainly be interesting to know the canonical basic sets for other choices of  $a$  and  $b$  as well, for example, the equal parameter case where  $a = b > 0$ , or the case where  $b = 0$  and  $a > 0$  (which is relevant to type  $D_n$ ). This will be discussed in Section 8.

## 7. THE FOCK SPACE AND CANONICAL BASES

We now take an excursion to another area of representation theory: the theory of canonical bases for highest weight modules of quantized enveloping algebras, as developed by Kashiwara and Lusztig. Of course, it is not the place here to give any reasonably detailed introduction (we refer the reader to Kashiwara [64] or Lusztig [72]), but what we can do is to explain some of the main ideas behind a deep combinatorial construction arising from that theory, namely, the *crystal graph* of a highest weight module. (In the following section, we will see applications to Iwahori–Hecke algebras.)

The starting point is the following idea. Recall that the simple  $\mathbb{Q}\mathfrak{S}_n$ -modules are parametrized by the set of all partitions of  $n$ . More generally, the simple modules of a Coxeter group of type  $B_n$  are parametrized by the set of all pairs of partitions such that the total sum of all parts equals  $n$ . Even more generally, the simple modules of the semidirect product

$$G_{r,n} := (\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n \quad (\text{where } r \geq 1)$$

are parametrized by the set  $\Pi_{r,n}$  of all  $r$ -tuples  $\lambda = (\lambda_{(1)}, \dots, \lambda_{(r)})$  where each  $\lambda_{(i)}$  is a partition of some  $a_i \geq 0$  and where  $n = a_1 + \dots + a_r$ . We shall now fix  $r$  and consider the simple modules for all the groups  $G_{r,n}$  ( $n \geq 1$ ) at the same time. For this purpose, let  $\mathfrak{F}_{r,n}$  be the  $\mathbb{C}$ -vector space with basis  $\Pi_{r,n}$ . The “Fock space” is defined as the direct sum of all these spaces:

$$\mathfrak{F}^{(r)} := \bigoplus_{n \geq 0} \mathfrak{F}_{r,n};$$

where  $\mathfrak{F}_{0,n}$  is the 1-dimensional space with basis  $\Pi_{0,n} = \{\emptyset := (\emptyset, \dots, \emptyset)\}$ . The point of collecting all these spaces into one object is that there are natural operators sending  $\mathfrak{F}_{r,n}$  into  $\mathfrak{F}_{r,n+1}$ , and vice versa. Indeed, the *branching rule* for the induction of representations from  $G_{r,n}$  to  $G_{r,n+1}$  gives rise to a linear map

$$\text{ind}: \mathfrak{F}^{(r)} \rightarrow \mathfrak{F}^{(r)}, \quad \lambda \mapsto \sum_{\mu} \mu \quad (\lambda \in \Pi_{r,n}),$$

where the sum runs over all  $\mu \in \Pi_{r,n+1}$  which can be obtained from  $\lambda$  by increasing exactly one part by 1. Similarly, the restriction of representations from  $G_{r,n}$  to  $G_{r,n-1}$  gives rise to a linear map

$$\text{res}: \mathfrak{F}^{(r)} \rightarrow \mathfrak{F}^{(r)}, \quad \lambda \mapsto \sum_{\mu} \mu \quad (\lambda \in \Pi_{r,n}),$$

where the sum runs over all  $\mu \in \Pi_{r,n-1}$  which can be obtained from  $\lambda$  by decreasing exactly one part by 1. Now, fixing a positive integer  $l \geq 2$  and a set of parameters

$$\mathbf{u} = \{u_1, \dots, u_r\} \quad \text{where} \quad u_i \in \mathbb{Z},$$

the operators  $\text{ind}$  and  $\text{res}$  can be refined into sums of linear operators

$$\text{ind} = \sum_{i=0}^{l-1} f_i \quad \text{and} \quad \text{res} = \sum_{i=0}^{l-1} e_i \quad \text{where} \quad e_i, f_i: \mathfrak{F}^{(r)} \rightarrow \mathfrak{F}^{(r)}.$$

The definition of these refined operators (which we will give further below) also has an interpretation in terms of representations; see [3, §12.1]. Now it is a remarkable fact that these operators turn out to satisfy the *Serre relations* for the affine Kac–Moody algebra  $\hat{\mathfrak{sl}}_l$ . This algebra is defined as

$$\mathfrak{g} = \hat{\mathfrak{sl}}_l := (\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{sl}_l) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where  $t$  is an indeterminate and  $\mathfrak{sl}_l$  is the usual Lie algebra of complex  $l \times l$ -matrices with trace zero and product  $[X, Y] = XY - YX$ . Thus, as a  $\mathbb{C}$ -vector space,  $\mathfrak{g}$  is spanned by  $\{t^n \otimes X \mid n \in \mathbb{Z}, X \in \mathfrak{sl}_l\} \cup \{c, d\}$ ; the Lie product is given by

$$\begin{aligned} [t^n \otimes X, t^m \otimes Y] &= t^{n+m} \otimes [X, Y] + \text{trace}(XY) n \delta_{0, n+m} c, \\ [c, t^n \otimes X] &= 0, \quad [d, t^n \otimes X] = n t^n \otimes X, \quad [c, d] = 0. \end{aligned}$$

In dealing with representations of  $\mathfrak{g}$ , it will be convenient to work with a presentation of the corresponding universal enveloping algebra  $U(\mathfrak{g})$ . We set

$$\begin{aligned} e_0 &:= t \otimes E_{l1}, & e_i &:= 1 \otimes E_{i, i+1} \quad (1 \leq i \leq l-1), \\ f_0 &:= t^{-1} \otimes E_{1l}, & f_i &:= 1 \otimes E_{i+1, i} \quad (1 \leq i \leq l-1), \\ h_0 &:= -\sum_{i=1}^{l-1} h_i + c, & h_i &:= 1 \otimes (E_{i, i} - E_{i+1, i+1}) \quad (1 \leq i \leq l-1), \end{aligned}$$

where the  $E_{i,j}$  are the usual matrix units in  $\mathfrak{sl}_l$ . Let

$$\mathfrak{h} := \langle d, h_0, h_1, \dots, h_{l-1} \rangle_{\mathbb{C}} \subseteq \mathfrak{g}$$

be the Cartan subalgebra. We have  $[h, h'] = 0$  for all  $h, h' \in \mathfrak{h}$ . For  $0 \leq i \leq l-1$ , we define linear forms  $\alpha_i \in \mathfrak{h}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$  by

$$\alpha_i(h_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i - j \equiv \pm 1 \pmod{l}, \\ 0 & \text{otherwise;} \end{cases}$$

and  $\alpha_i(d) = \delta_{i0}$ . By the classification in Kac [63], the matrix

$$A = (\alpha_i(h_j))_{0 \leq i, j \leq l-1}$$

is the Cartan matrix of affine type  $A_{l-1}^{(1)}$ . With this notation,  $U(\mathfrak{g})$  is the associative algebra (with 1) generated by elements

$$\{e_i, f_i, h_i \mid 0 \leq i \leq l-1\} \cup \{d\},$$

and subject to the following defining relations:

$$\begin{aligned} h_j e_i - e_i h_j &= \alpha_i(h_j) e_i, & h_j f_i - f_i h_j &= -\alpha_i(h_j) f_i, \\ d e_i - e_i d &= \delta_{i0} e_i, & d f_i - f_i d &= -\delta_{i0} f_i, \\ e_i f_j - f_j e_i &= \delta_{ij} h_i, & h_i h_j &= h_j h_i, & h_i d &= d h_i, \\ e_i e_j &= e_j e_i & \text{and} & & f_i f_j &= f_j f_i & \text{(if } i - j \not\equiv \pm 1 \pmod{l}). \end{aligned}$$

Furthermore, if  $l \geq 3$  and  $i - j \equiv \pm 1 \pmod{l}$ , then

$$e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0 \quad \text{and} \quad f_i^2 f_j - 2f_i f_j f_i + f_j f_i^2 = 0;$$

if  $l = 2$  and  $i \neq j$ , then

$$\begin{aligned} e_i^3 e_j - 3(e_i^2 e_j e_i + e_i e_j e_i^2) - e_j e_i^3 &= 0, \\ f_i^3 f_j - 3(f_i^2 f_j f_i + f_i f_j f_i^2) - f_j f_i^3 &= 0. \end{aligned}$$

For a sketch of proof and further references, see Ariki [3, §3.2]. We will now define a linear action of  $U(\mathfrak{g})$  on  $\mathfrak{F}^{(r)}$ . For this purpose, we need some further notation. Let  $\lambda = (\lambda_{(1)}, \dots, \lambda_{(r)}) \in \Pi_{r,n}$  and write

$$\lambda_{(c)} = (\lambda_{(c),1} \geq \lambda_{(c),2} \geq \dots \geq 0) \quad \text{for} \quad c = 1, \dots, r.$$

The diagram of  $\lambda$  is defined as the set

$$[\lambda] := \{(a, b, c) \mid 1 \leq c \leq r, 1 \leq b \leq \lambda_{(c),a} \text{ for } a = 1, 2, \dots\}.$$

For any “node”  $\gamma = (a, b, c) \in [\lambda]$ , we set

$$\text{res}_l(\gamma) := (b - a + u_c) \pmod{l}$$

and call this the  $l$ -residue of  $\gamma$  with respect to the parameters  $\mathbf{u}$ . If  $\text{res}_l(\gamma) = i$ , we say that  $\gamma$  is an  $i$ -node of  $\lambda$ . Let

$$W_i(\lambda) := \text{number of } i\text{-nodes of } \lambda.$$

Now suppose that  $\lambda \in \Pi_{r,n}$  and  $\mu \in \Pi_{r,n+1}$  for some  $n \geq 0$ . We write

$$\gamma = \mu / \lambda \quad \text{if} \quad [\lambda] \subset [\mu] \quad \text{and} \quad [\mu] = [\lambda] \cup \{\gamma\};$$

Then we call  $\gamma$  an addable node for  $\lambda$  or a removable node for  $\mu$ . Let

$$A_i(\lambda) := \text{set of addable } i\text{-nodes for } \lambda,$$

$$R_i(\mu) := \text{set of removable } i\text{-nodes for } \mu.$$

Now let  $0 \leq i \leq l - 1$ . We define a linear operation of  $e_i$  on  $\mathfrak{F}^{(r)}$  by

$$e_i \cdot \lambda = \sum_{\mu} \mu \quad (\lambda \in \Pi_{r,n})$$

where the sum runs over all  $\mu \in \Pi_{r,n-1}$  such that  $[\mu] \subset [\lambda]$  and  $\text{res}_l(\lambda / \mu) \equiv i \pmod{l}$ . Similarly, we define a linear operation of  $f_i$  on  $\mathfrak{F}^{(r)}$  by

$$f_i \cdot \lambda = \sum_{\mu} \mu \quad (\lambda \in \Pi_{r,n})$$

where the sum runs over all  $\mu \in \Pi_{r,n+1}$  such that  $[\lambda] \subset [\mu]$  and  $\text{res}_l(\mu / \lambda) \equiv i \pmod{l}$ . Note that, indeed, we have  $\text{res} = \sum_i e_i$  and  $\text{ind} = \sum_i f_i$ .



Next, we define a linear operation of  $h_i$  and  $d$  on  $\mathfrak{F}^{(r)}$  by

$$h_i.\lambda = N_i(\lambda)\lambda \quad \text{and} \quad d.\lambda = -W_0(\lambda)\lambda,$$

where  $N_i(\lambda) = |A_i(\lambda)| - |R_i(\lambda)|$ . The following result (extended to a “quantized” version) is originally due to Hayashi for  $r = 1$ ; see [3, Chap. 10] for the proof and further historical remarks.

**Proposition 7.1** (See Ariki [3, Lemma 13.35]). *Via the above maps,  $\mathfrak{F}^{(r)}$  becomes an integrable  $U(\mathfrak{g})$ -module. Let  $M^{(r)}(\underline{\varnothing})$  be the submodule generated by  $\underline{\varnothing} \in \mathfrak{F}_{r,0}$ . Then  $M^{(r)}(\underline{\varnothing})$  is a highest weight module with highest weight given by  $\sum_{i=1}^r \Lambda_{\gamma_i}$  where  $\gamma_i \in \{0, 1, \dots, l-1\}$  is such that  $\gamma_i \equiv u_i \pmod{l}$ .*

Here, the fundamental weights  $\Lambda_0, \Lambda_1, \dots, \Lambda_{l-1}$  for  $\mathfrak{g}$  are the elements of  $\mathfrak{h}^*$  defined by  $\Lambda_i(h_j) = \delta_{ij}$  and  $\Lambda_i(d) = 0$ . We refer to Kac [63] for the general theory of integrable modules for Kac–Moody algebras.

In order to speak about the *canonical basis* of  $M^{(r)}(\underline{\varnothing})$ , we need to consider a “deformed” version of  $U(\mathfrak{g})$ , that is, the quantized universal enveloping algebra  $U_v(\mathfrak{g})$ , where  $v$  is an indeterminate. Following Ariki [3, §3.3],  $U_v(\mathfrak{g})$  is the associative  $\mathbb{C}(v)$ -algebra (with 1), generated by elements

$$\{E_i, F_i, K_i, K_i^{-1} \mid 0 \leq i \leq l-1\} \cup \{D, D^{-1}\},$$

subject to the following defining relations:

$$\begin{aligned} K_j E_i K_j^{-1} &= v^{\alpha_i(h_j)} E_i, & K_j F_i K_j^{-1} &= v^{-\alpha_i(h_j)} F_i, \\ D E_i D^{-1} &= v^{\delta_{0i}} E_i, & D F_i D^{-1} &= v^{-\delta_{0i}} F_i, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{v - v^{-1}}, & K_i K_j &= K_j K_i, \quad D K_i = K_i D, \\ E_i E_j &= E_j E_i \quad \text{and} \quad F_i F_j = F_j F_i && \text{(if } i - j \not\equiv \pm 1 \pmod{l}). \end{aligned}$$

Furthermore, if  $l \geq 3$  and  $i - j \equiv \pm 1 \pmod{l}$ , then

$$\begin{aligned} E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 &= 0, \\ F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 &= 0; \end{aligned}$$

if  $l = 2$  and  $i \neq j$ , then

$$\begin{aligned} E_i^3 E_j - (v^2 + 1 + v^{-2}) (E_i^2 E_j E_i + E_i E_j E_i^2) - E_j E_i^3 &= 0, \\ F_i^3 F_j - (v^2 + 1 + v^{-2}) (F_i^2 F_j F_i + F_i F_j F_i^2) - F_j F_i^3 &= 0. \end{aligned}$$

We can extend the action of  $U(\mathfrak{g})$  on  $\mathfrak{F}^{(r)}$  to an action of  $U_v(\mathfrak{g})$  on

$$\mathfrak{F}_v^{(r)} := \mathbb{C}(v) \otimes_{\mathbb{C}} \mathfrak{F}^{(r)}.$$

This will depend on the choice of a total order on nodes.

**Definition 7.2** (Foda et al. [25, p. 331]). We say that the node  $\gamma = (a, b, c)$  is “above” the node  $\gamma' = (a', b', c')$  if

$$b - a + u_c < b' - a' + u_{c'} \quad \text{or if} \quad b - a + u_c = b' - a' + u_{c'} \quad \text{and} \quad c' < c.$$

Now let  $0 \leq i \leq l-1$ . We define a linear operation of  $E_i$  on  $\mathfrak{F}_v^{(r)}$  by

$$E_i.\lambda = \sum_{\mu} v^{-N_i^a(\lambda/\mu)} \mu \quad (\lambda \in \Lambda_{r,n})$$

where the sum runs over all  $\mu \in \Pi_{r,n-1}$  such that  $[\lambda] = [\mu] \cup \{\gamma\}$ ,  $\text{res}_l(\gamma) \equiv i \pmod l$  and

$$N_i^a(\lambda/\mu) = |\{\gamma' \in A_i(\mu) \mid \gamma' \text{ above } \gamma\}| - |\{\gamma' \in R_i(\lambda) \mid \gamma' \text{ above } \gamma\}|.$$

Similarly, we define a linear operation of  $F_i$  on  $\mathfrak{F}_v^{(r)}$  by

$$F_i.\lambda = \sum_{\mu} v^{N_i^b(\mu/\lambda)} \mu \quad (\lambda \in \Lambda_{r,n})$$

where the sum runs over all  $\mu \in \Pi_{r,n+1}$  such that  $[\mu] = [\lambda] \cup \{\gamma\}$ ,  $\text{res}_l(\gamma) \equiv i \pmod l$  and

$$N_i^b(\mu/\lambda) = |\{\gamma' \in A_i(\lambda) \mid \gamma \text{ above } \gamma'\}| - |\{\gamma' \in R_i(\mu) \mid \gamma \text{ above } \gamma'\}|.$$

Next, we define linear operations of  $K_i$  and  $D$  on  $\mathfrak{F}_v^{(r)}$  by

$$K_i.\lambda = v^{N_i^b(\lambda)} \lambda \quad \text{and} \quad D.\lambda = v^{-W_0(\lambda)} \lambda.$$

**Theorem 7.3** (Jimbo et al. [62], Foda et al. [25], Uglov [81]). *Via the above maps,  $\mathfrak{F}_v^{(r)}$  becomes an integrable  $U_v(\mathfrak{g})$ -module. The submodule  $M_v^{(r)}(\underline{\varnothing})$  generated by  $\underline{\varnothing}$  is a highest weight module with highest weight  $\sum_{i=1}^r \Lambda_{\gamma_i}$ .*

*Remark 7.4.* The above definition of a  $v$ -deformed action of  $U(\mathfrak{g})$  on  $\mathfrak{F}^{(r)}$  is not the only possible one. Ariki [3, Theorem 10.10] considers an action of  $U_v(\mathfrak{g})$  on  $\mathfrak{F}_v^{(r)}$  which is given by exactly the same formulae as above, but where the exponents  $N_i^a$  and  $N_i^b$  are computed with respect to the following order on nodes: let  $\gamma = (a, b, c)$  and  $\gamma' = (a', b', c')$ ; then

$$\gamma \prec \gamma' \stackrel{\text{def}}{\iff} c' < c \quad \text{or if} \quad c = c' \text{ and } a' < a.$$

Note that  $\prec$  does not depend on the parameters  $\mathbf{u}$ ! A relation between this order and the one in Definition 7.2 can be established as follows. Suppose we are only interested in  $\bigoplus_{0 \leq k \leq n} \mathfrak{F}_{r,k}$  for some fixed  $n$ . Then choose the integers  $u_1, \dots, u_r$  such that

$$u_1 > u_2 > \dots > u_r > 0 \quad \text{where} \quad u_i - u_{i+1} > n - 1 \text{ for all } i.$$

Let  $\gamma = (a, b, c)$  and  $\gamma' = (a', b', c')$  for some partitions in  $\Pi_{r,k}$  where  $0 \leq k \leq n$ . One easily checks that  $\gamma$  lies above  $\gamma'$  if and only if  $\gamma \prec \gamma'$ .

Since  $M_v^{(r)}(\underline{\varnothing})$  is an integrable highest weight module, the general theory of Kashiwara and Lusztig provides us with a canonical basis of  $M_v^{(r)}(\underline{\varnothing})$ . In Lusztig's setting, this basis is given by

$$\mathbf{B}(\underline{\varnothing}) = \{b.\underline{\varnothing} \mid b \in \mathbf{B}\} \setminus \{0\},$$

where  $\mathbf{B}$  is the *canonical basis* of the subalgebra  $U_v^-(\mathfrak{g}) \subseteq U_v(\mathfrak{g})$  generated by  $F_0, F_1, \dots, F_{l-1}$ ; see [3, Theorem 7.3, Prop. 9.1] and the references there.

Setting  $v = 1$ , we obtain the canonical basis, denoted  $\mathbf{B}_1(\underline{\emptyset})$ , of the  $U(\mathfrak{g})$ -module  $M^{(r)}(\underline{\emptyset})$ .

In order to make an efficient use of this result, we need a good parametrization of the canonical basis. This is provided by Kashiwara's *crystal basis*, which is obtained by specializing  $v$  to 0. A modified action of the generators  $E_i$  and  $F_i$  sends the crystal basis to itself, and this gives the crystal basis the structure of a colored oriented graph. In this way, many properties of the representation  $M^{(r)}(\underline{\emptyset})$  can be studied using combinatorial properties of that graph. Let us briefly discuss how all this works, where we follow Kashiwara [64]. The starting point is the observation that the subalgebra

$$U_{v,i} := \langle E_i, F_i, K_i^{\pm 1} \rangle \subseteq U_v(\mathfrak{g}) \quad (0 \leq i \leq l-1)$$

is isomorphic to the quantized enveloping algebra of the 3-dimensional Lie algebra  $\mathfrak{sl}_2$ . We have the relations:

$$\begin{aligned} K_i E_i K_i^{-1} &= v^2 E_i, & K_i F_i K_i^{-1} &= v^{-2} F_i, \\ E_i F_i - F_i E_i &= \frac{K_i - K_i^{-1}}{v - v^{-1}}. \end{aligned}$$

We will study properties of a module for  $U_v(\mathfrak{g})$  by restricting the action to  $U_{v,i}$  and using known properties of representations of  $U_{v,i} \cong U_v(\mathfrak{sl}_2)$ . Recall that, for any  $m \geq 0$ , there are precisely two simple  $U_{v,i}$ -modules of dimension  $m+1$  (up to isomorphism), denoted by  $V_m^{\pm}$ . We have that  $V_m^{\pm}$  is a cyclic module generated by a vector  $u_0$  such that

$$E_i \cdot u_0 = 0 \quad \text{and} \quad K_i \cdot u_0 = \pm v^m u_0.$$

See [64, §1.3] for further details on the construction of  $V_m^{\pm}$ .

Now let  $M$  be an integrable  $U_v(\mathfrak{g})$ -module. We do not need to recall the precise definition here, but what is important is the fact that, regarded as a  $U_{v,i}$ -module,  $M$  can be written as a direct sum of modules of the form  $V_m^{\pm}$ ; see [64, 1.4.1 and 3.3.1]. The explicit knowledge of the action of  $U_{v,i}$  on  $V_m^{\pm}$  implies that we have a direct sum decomposition

$$M = \bigoplus_{n \in \mathbb{Z}} M_n \quad \text{where} \quad M_n := \{u \in M \mid K_i \cdot u = v^n u\};$$

see [64, §1.4]. For a fixed  $n \in \mathbb{Z}$ , we have  $\dim M_n < \infty$  and any  $u \in M_n$  has a unique expression

$$u = \sum_{m \geq 0, -n} \frac{1}{[m]!} F_i^m \cdot u_m \quad \text{where } u_m \in M_{n+2m} \text{ is such that } E_i \cdot u_m = 0.$$

Here,  $[0]! = 1$  and  $[m]! = [m-1]!(v^m - v^{-m})/(v - v^{-1})$  for  $m > 0$ . For these facts, see [64, §2.2].

**Definition 7.5** (Kashiwara; see [64, 2.2.4]). For  $0 \leq i \leq l-1$ , we define linear maps  $\tilde{E}_i, \tilde{F}_i: M \rightarrow M$  by

$$\begin{aligned}\tilde{E}_i.u &= \sum_{m \geq 1, -n} \frac{1}{[m-1]!} F_i^{m-1}.u_m \quad (u \in M_n), \\ \tilde{F}_i.u &= \sum_{m \geq 0, -n} \frac{1}{[m+1]!} F_i^{m+1}.u_m \quad (u \in M_n).\end{aligned}$$

Next, let  $R \subseteq \mathbb{C}(v)$  be the local ring of all rational functions with no pole at  $v = 0$ . A *local basis* of  $M$  is a pair  $(\mathcal{L}, \mathcal{B})$  where  $\mathcal{L}$  is an  $R$ -submodule of  $M$  such that  $M$  is a free as an  $R$ -module,  $\mathbb{C}(v) \otimes_R \mathcal{L} = M$ , and  $\mathcal{B}$  is a basis of the  $\mathbb{C}$ -vector space  $\mathcal{L}/v\mathcal{L}$ . Note that a complete set of representatives of  $\mathcal{B}$  in  $\mathcal{L}$  forms a  $\mathbb{C}(v)$ -basis of  $M$ . A *crystal basis* for  $M$  is a local basis  $(\mathcal{L}, \mathcal{B})$  satisfying certain additional conditions. These conditions include the following (see [64, Theorem 4.1.2]):

- we have  $\tilde{E}_i.\mathcal{L} \subseteq \mathcal{L}$  and  $\tilde{F}_i.\mathcal{L} \subseteq \mathcal{L}$  for  $0 \leq i \leq l-1$ ; hence, we obtain actions of  $\tilde{E}_i$  and  $\tilde{F}_i$  on  $\mathcal{L}/v\mathcal{L}$  which we denote by the same symbols;
- we have  $\tilde{E}_i.\mathcal{B} \subseteq \mathcal{B} \cup \{0\}$  and  $\tilde{F}_i.\mathcal{B} \subseteq \mathcal{B} \cup \{0\}$  for  $0 \leq i \leq l-1$ ;
- for  $b, b' \in \mathcal{B}$  and  $0 \leq i \leq n-1$ , we have  $b' = \tilde{F}_i.b \Leftrightarrow \tilde{E}_i.b' = b$ .

Thus, we can define a graph with vertices indexed by the elements of  $\mathcal{B}$ ; for  $b \neq b'$  in  $\mathcal{B}$  and  $0 \leq i \leq n-1$ , we have a colored oriented edge

$$b \xrightarrow{i} b' \quad \text{if and only if} \quad b' = \tilde{F}_i.b.$$

This graph is the *crystal graph* of  $\mathcal{B}$ . By [3, Chap. 9], if  $M$  is an integrable highest weight module, then  $M$  has a crystal basis which is unique up to scalar multiples; furthermore, using Lusztig's canonical basis  $\mathbf{B}$  of  $U_v^-(\mathfrak{g})$ , we obtain a crystal basis of  $M$  by setting

$$\mathcal{L} := \sum_{b \in \mathbf{B}} Rb.u_0 \quad \text{and} \quad \mathcal{B} := \{b.u_0 + v\mathcal{L} \mid b \in \mathbf{B}\} \setminus \{0\},$$

where  $u_0 \in M$  is a highest weight vector.

Now the problem is to obtain an explicit description of the crystal graph for our highest weight module  $M_v^{(r)}(\varnothing)$ . For this purpose, we first describe a crystal basis of the integrable module  $\mathfrak{F}_v^{(r)}$ . The corresponding graph will have vertices labelled by the set  $\Pi_r := \bigcup_{n \geq 0} \Pi_{r,n}$  and it will have the property that the connected component of  $\varnothing$  is the crystal graph of  $M_v^{(r)}(\varnothing)$ .

**Theorem 7.6** (Jimbo et al. [62], Foda et al. [25], Uglov [81]). *We set*

$$\mathcal{L}_r := \sum_{\lambda \in \Pi_r} R\lambda \subseteq \mathfrak{F}_v^{(r)} \quad \text{and} \quad \mathcal{B}_r := \{b_\lambda \mid \lambda \in \Pi_r\},$$

where  $b_\lambda$  denotes the image of  $\lambda$  in  $v\mathcal{L}_r$ . Then  $(\mathcal{L}_r, \mathcal{B}_r)$  is a crystal basis for  $\mathfrak{F}_v^{(r)}$ . Given  $\lambda, \mu \in \Pi_r$  and  $i \in \{0, 1, \dots, l-1\}$ , we have  $b_\lambda \xrightarrow{i} b_\mu$  if and only if  $\mu$  is obtained from  $\lambda$  by adding a so-called “good”  $i$ -node.

The “good” nodes are defined as follows. Let  $\lambda \in \Pi_r$  and let  $\gamma$  be an  $i$ -node of  $\lambda$ . We say that  $\gamma$  is a *normal* node if, whenever  $\gamma'$  is an  $i$ -node of  $\lambda$  below  $\gamma$ , there are strictly more removable  $i$ -nodes between  $\gamma'$  and  $\gamma$  than there are addable  $i$ -nodes between  $\gamma'$  and  $\gamma$ . If  $\gamma$  is a highest normal  $i$ -node of  $\lambda$ , then  $\gamma$  is called a *good* node. Note that these notions heavily depend on the definition of what it means for one node to be “above” another node. These definitions (for  $r = 1$ ) first appeared in the work of Kleshchev [65] on the modular branching rule for the symmetric group; see also the discussion of these results in [66, §2].

For any  $n \geq 0$ , we define a subset  $\Lambda_{r,n}^{(\mathbf{u})} \subseteq \Pi_{r,n}$  recursively as follows. We set  $\Lambda_{r,0}^{(\mathbf{u})} = \{\emptyset\}$ . For  $n \geq 1$ , the set  $\Lambda_{r,n}^{(\mathbf{u})}$  is constructed as follows.

- (1) We have  $\emptyset \in \Lambda_{r,n}^{(\mathbf{u})}$ ;
- (2) Let  $\lambda \in \Pi_{r,n}$ . Then  $\lambda$  belongs to  $\Lambda_{r,n}^{(\mathbf{u})}$  if and only if  $\lambda/\mu = \gamma$  where  $\mu \in \Lambda_{r,n-1}^{(\mathbf{u})}$  and  $\gamma$  is a good  $i$ -node of  $\lambda$  for some  $i \in \{0, 1, \dots, l-1\}$ .

Thus, the set  $\Lambda_r^{(\mathbf{u})} := \bigcup_{n \geq 0} \Lambda_{r,n}^{(\mathbf{u})}$  labels the vertices in the connected component containing  $b_{\emptyset}$  of the crystal graph of  $\mathfrak{F}_v^{(r)}$ . Hence, by general results on crystal bases (see the “unicity theorem” in [64, 4.1.5]), we have:

**Corollary 7.7.** *The crystal graph of  $M_v^{(r)}(\emptyset)$  has vertices labelled by the elements in  $\Lambda_r^{(\mathbf{u})}$ . The edges  $b_{\lambda} \xrightarrow{i} b_{\mu}$  are given as in Theorem 7.6.*

The above result provides a purely combinatorial description of the crystal graph of  $M_v^{(r)}(\emptyset)$ . An example is given in Table 2. In that example, we have

$$\Lambda_{2,3}^{(0,1)} = \{((3), \emptyset), ((2), (1)), ((1), (2)), (\emptyset, (3))\}.$$

If we had used the ordering of nodes  $\prec$  in Remark 7.4, we would obtain a quite different labelling of the vertices and edges in the crystal graph; in that case,  $\Lambda_{2,3}^{(0,1)}$  would have to be replaced by the set (see [55, §1.3]):

$$\{((3), \emptyset), ((2, 1), \emptyset), ((1), (2)), ((2), (1))\}.$$

In Section 8, we will see interpretations of these different labellings in terms of modular representations of Iwahori–Hecke algebras of type  $B_n$ .

**7.8. FLOTW-partitions.** Assume that the parameters in  $\mathbf{u}$  satisfy the condition

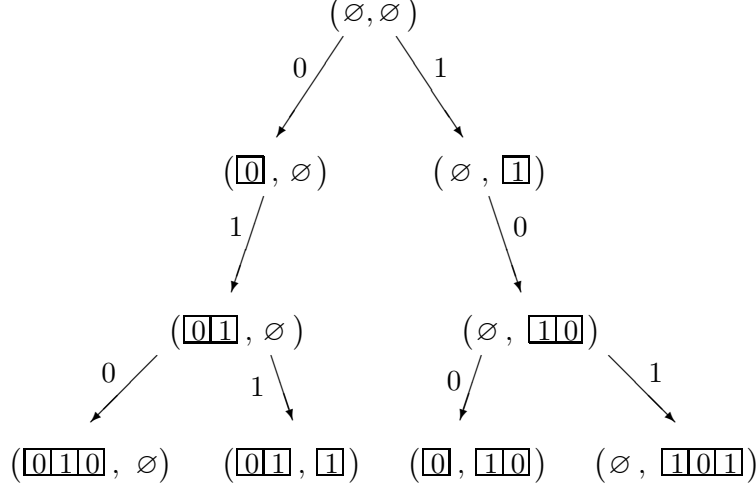
$$0 \leq u_1 \leq u_2 \leq \dots \leq u_r \leq l-1.$$

Then it is shown in Foda et al. [25, 2.11] that  $\lambda \in \Pi_{r,n}$  belongs to  $\Lambda_{r,n}^{(\mathbf{u})}$  if and only if the following conditions are satisfied:

- (a) For all  $1 \leq j \leq r-1$  and  $i = 1, 2, \dots$ , we have:

$$\lambda_{(j+1),i} \geq \lambda_{(j),i+u_{j+1}-u_j} \quad \text{and} \quad \lambda_{(1),i} \geq \lambda_{(r),i+l+u_1-u_r};$$

- (b) for all  $k > 0$ , among the residues appearing at the right ends of the rows of  $[\lambda]$  of length  $k$ , at least one element of  $\{0, 1, \dots, l-1\}$  does not occur.

TABLE 2. Part of the crystal graph for  $l = r = 2$ ,  $\mathbf{u} = (0, 1)$ 

(The numbers inscribed in the boxes of the diagrams are the  $l$ -residues.)

Note that this provides a non-recursive description of the elements of  $\Lambda_r^{(\mathbf{u})}$ .

## 8. THE THEOREMS OF ARIKI AND JACON

The aim of this section is to explain the applications of the results on the canonical basis of the Fock space  $\mathfrak{F}^{(r)}$  to the problem of parametrizing the simple modules of non-semisimple Iwahori–Hecke algebras. These results actually hold for a wider class of “Hecke algebras”, which we now introduce.

**8.1. Ariki–Koike algebras.** Let  $k$  be an algebraically closed field and let  $\zeta_l$  be an element of order  $l \geq 2$  in  $k^\times$ . Let  $r, n \geq 1$  and fix parameters

$$\mathbf{u} = (u_1, \dots, u_r) \quad \text{where} \quad u_i \in \mathbb{Z}.$$

Having fixed these data, we let  $\mathbf{H}_{r,n}^{(\mathbf{u})}$  be the associative  $k$ -algebra (with 1), with generators  $S_0, S_1, \dots, S_{n-1}$  and defining relations as follows:

$$\begin{aligned}
 S_0 S_1 S_0 S_1 &= S_1 S_0 S_1 S_0 \quad \text{and} \quad S_0 S_i = S_i S_0 \quad (\text{for } i > 1), \\
 S_i S_j &= S_j S_i \quad (\text{if } |i - j| > 1), \\
 S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1} \quad (\text{for } 1 \leq i \leq n - 2), \\
 (S_0 - \zeta_l^{u_1})(S_0 - \zeta_l^{u_2}) \cdots (S_0 - \zeta_l^{u_r}) &= 0, \\
 (S_i - \zeta_l)(S_i + 1) &= 0 \quad \text{for } 1 \leq i \leq n - 1.
 \end{aligned}$$

This algebra can be seen as an Iwahori–Hecke algebra associated with the group  $G_{r,n}$  (which is a Coxeter group only for  $r = 1, 2$ ); see Ariki [3, Chap. 13] and Broué–Malle [10] for further details and motivations for studying this class of algebras.

Here, we will be mostly interested in the case  $r = 2$ , when  $\mathbf{H}_{2,n}^{(\mathbf{u})}$  can be identified with an Iwahori–Hecke algebra of type  $B_n$ . Indeed, let  $\zeta_{2l} \in k^\times$  be a square root of  $\zeta_l$  such that

$$\zeta_{2l}^2 = \zeta_l \quad \text{and} \quad \zeta_{2l}^l = -1.$$

Then, setting  $T := \zeta_{2l}^{l-2u_2} S_0$ , we obtain

$$\begin{aligned} T^2 &= \zeta_{2l}^f + (\zeta_{2l}^f - 1)T & \text{where} \quad f &= l + 2(u_1 - u_2), \\ S_i^2 &= \zeta_{2l}^2 + (\zeta_{2l}^2 - 1)S_i & \text{for } 1 \leq i \leq n-1. \end{aligned}$$

Thus, the map  $T \mapsto T_t$ ,  $S_i \mapsto T_{s_i}$  defines an isomorphism of  $k$ -algebras

$$\mathbf{H}_{2,n}^{(\mathbf{u})} \cong \mathbf{H}_{k,\xi} = k \otimes \mathbf{H}_A(W_n, L), \quad \xi = \zeta_{2l},$$

where  $W_n$  is defined as in Example 4.9 and  $L$  is a weight function such that

$$L(t) \equiv l + 2(u_2 - u_1) \pmod{2l} \quad \text{and} \quad L(s_i) \equiv 2 \pmod{2l}.$$

As mentioned at the beginning of the previous section, we have a labelling of the simple  $\mathbb{C}G_{r,n}$ -modules by  $\Pi_{r,n}$ . We write this as

$$\text{Irr}_{\mathbb{C}}(G_{r,n}) = \{E^\lambda \mid \lambda \in \Pi_{r,n}\}.$$

Now Dipper–James–Mathas [22, §3] have generalized the theory of *Specht modules* for Iwahori–Hecke algebra of the symmetric group to  $\mathbf{H}_{r,n}^{(\mathbf{u})}$ ; see Example 6.9 for the case  $r = 2$  and also Graham–Lehrer [49]. Thus, for any  $\lambda \in \Pi_{r,n}$ , there is a *Specht module*  $S^\lambda \in \mathbf{H}_{r,n}^{(\mathbf{u})}$ . Each  $S^\lambda$  carries a symmetric bilinear form and, taking quotients by the radical, we obtain a collection of modules  $D^\lambda$ . As before, we set  $\Lambda_{r,n}^\circ := \{\lambda \vdash n \mid D^\lambda \neq \{0\}\}$ . Then, by [22, Theorem 3.30], we have

$$\text{Irr}(\mathbf{H}_{r,n}^{(\mathbf{u})}) = \{D^\lambda \mid \lambda \in \Lambda_{r,n}^\circ\}.$$

Furthermore, the entries of the decomposition matrix

$$D = ([S^\lambda : D^\mu])_{\lambda \in \Pi_{r,n}, \mu \in \Lambda_{r,n}^\circ}$$

satisfy conditions analogous to those in Example 6.9, where one has to consider the dominance order on  $r$ -tuples of partitions defined in [22, 3.11].

We are now ready to state Ariki’s theorem which establishes the link to the Fock space  $\mathfrak{F}^{(r)}$ . For any  $\mu \in \Lambda_{r,n}^\circ$ , we define an element of  $\mathfrak{F}^{(r)}$  by

$$P_\mu := \sum_{\lambda \vdash n} [S^\lambda : D^\mu] \lambda \in \mathfrak{F}_{r,n}.$$

Then we consider the subspace of  $\mathfrak{F}^{(r)}$  generated by all these elements:

$$\mathcal{M}^{(r)} := \langle P_\mu \mid \mu \in \Lambda_{r,n}^\circ \text{ for some } n \geq 0 \rangle_{\mathbb{C}} \subseteq \mathfrak{F}^{(r)},$$

where  $P_\emptyset = \emptyset \in \mathfrak{F}_{r,0}$ .

**Theorem 8.2** (Ariki [3, Theorem 12.5]). *Assume that the characteristic of  $k$  is zero. Then we have*

$$\mathcal{M}^{(r)} = M^{(r)}(\varnothing) \quad \text{and} \quad \mathbf{B}_1(\varnothing) = \{P_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \Lambda_{r,n}^{\circ} \text{ for some } n \geq 0\},$$

where  $M^{(r)}(\varnothing)$  is the highest weight module for  $U(\hat{\mathfrak{sl}}_l)$  as in Proposition 7.1 and  $\mathbf{B}_1(\varnothing)$  is the Kashiwara–Lusztig canonical basis of  $M^{(r)}(\varnothing)$  (at  $v = 1$ ).

The importance of this result for the modular representation theory of Iwahori–Hecke algebras (and for finite groups of Lie type, via the results in Section 1 and 2) can hardly be overestimated. There are efficient and purely combinatorial algorithms for computing the canonical basis; see Lascoux–Leclerc–Thibon [66] (the “LLT algorithm”) for the case  $r = 1$  and Jacon [58] for  $r \geq 2$ . Hence, if  $k$  has characteristic zero, the above result shows that these algorithms compute the decomposition numbers of  $\mathbf{H}_{r,n}^{(\mathbf{u})}$ .

If the characteristic of  $k$  is not zero, the elements  $P_{\boldsymbol{\mu}}$  will no longer coincide with the elements of the canonical basis. But the first part of the above statement remains valid:

**Theorem 8.3** (Ariki–Mathas [4]). *We have  $\mathcal{M}^{(r)} = M^{(r)}(\varnothing)$  and*

$$|\text{Irr}(\mathbf{H}_{r,n}^{(\mathbf{u})})| = \dim_{\mathbb{C}}(\mathfrak{F}_{r,n} \cap M^{(r)}(\varnothing)).$$

*In particular, the number of simple modules of  $\mathbf{H}_{r,n}^{(\mathbf{u})}$  only depends on  $l$  and on the congruence classes modulo  $l$  of the numbers  $u_1, \dots, u_r$ .*

Using the above results, Ariki identified the indexing set  $\Lambda_{r,n}^{\circ}$  arising from the theory of Specht modules:

**Theorem 8.4** (Ariki [2, Theorem 4.3]). *We have  $\Lambda_{r,n}^{\circ} = \Lambda_{r,n}^{(\mathbf{u})}$ , where  $\mathbf{u} = (u_1, \dots, u_r)$  is chosen such that  $u_i - u_{i+1} > n - 1$  for all  $i$ .*

The condition  $u_i - u_{i+1} > n - 1$  really means that we are working with the order  $\prec$  on nodes defined in Remark 7.4. Then the set  $\Lambda_{r,n}^{(\mathbf{u})}$  is precisely the set of so-called *Kleshchev bipartitions*; see also Ariki [3, §12.2].

Now the algebra  $\mathbf{H}_{r,n}^{(\mathbf{u})}$  also is symmetric, with a trace function  $\tau: \mathbf{H}_{r,n}^{(\mathbf{u})} \rightarrow k$  satisfying properties analogous to those for Iwahori–Hecke algebras of finite Coxeter groups; see Bremke–Malle [9], Malle–Mathas [75]. Working with a suitable generic version of  $\mathbf{H}_{r,n}^{(\mathbf{u})}$ , we have corresponding Laurent polynomials  $\mathbf{c}_{E\lambda}$  for any  $\lambda \in \Pi_{r,n}$ . Explicit combinatorial formulae for  $\mathbf{c}_{E\lambda}$ , generalizing those in Example 4.9 for type  $B_n$ , are obtained by Geck–Iancu–Malle [44] and, independently, by Mathas [76] (proving a conjecture of Malle [74]).

Thus, it makes sense to consider the existence of “canonical basic sets” for  $\mathbf{H}_{r,n}^{(\mathbf{u})}$ , as in Definition 4.13. Recently, Jacon [55], [59] has shown that such canonical basic sets indeed exist for  $\mathbf{H}_{r,n}^{(\mathbf{u})}$ . (His results still rely on Ariki’s Theorem 8.2.) These are labelled by sets  $\Lambda_{r,n}^{(\mathbf{u})}$  for various choices of  $\mathbf{u}$  where the non-recursive description of Foda et al. in (7.8) applies.



Let us discuss some applications of Jacon's results to the case  $r = 2$ . Then  $\mathbf{H}_{2,n}^{(\mathbf{u})}$  can be identified with an Iwahori–Hecke algebra of type  $B_n$ , as explained above. So let  $W_n$  be a Coxeter group of type  $B_n$ , with weight function  $L: W_n \rightarrow \mathbb{N}$  determined by two integers  $a, b \geq 0$  as in Example 4.9:

$$\begin{array}{ccccccc} B_n & & t & s_1 & s_2 & \dots & s_{n-1} \\ L : & & \bullet & \bullet & \bullet & \dots & \bullet \\ & & b & a & a & \dots & a \end{array}$$

Let  $\theta: A \rightarrow k$  be a specialization into a field  $k$  of characteristic  $\neq 2$ ; let  $\xi = \theta(v^2)$ . In Example 6.9, we have already seen a convenient parametrization of the simple modules of  $\mathbf{H}_{k,\xi}$  in the cases where  $f_n(a, b) \neq 0$  or  $\xi^a = 1$ . So let us now assume that

$$f_n(a, b) = 0 \quad \text{and} \quad \xi^a \neq 1.$$

Then we have

$$\xi^{b+ad} = -1 \quad \text{for some } d \in \mathbb{Z} \text{ such that } |d| < n-1.$$

In particular,  $\xi$  is a root of unity of even order. Let  $l \geq 2$  be the multiplicative order of  $\xi^a$  and set  $\zeta_l := \xi^a$ . Then  $\xi^b = -\xi^{-ad} = -\zeta_l^{-d}$ . Let  $\zeta_{2l} \in k$  be a square root of  $\zeta_l$  such that  $\zeta_{2l}^l = -1$ . Thus, we have the quadratic relations

$$T_t^2 = \zeta_{2l}^{l-2d} + (\zeta_{2l}^{l-2d} - 1)T_t \quad \text{and} \quad T_{s_i} = \zeta_{2l}^2 + (\zeta_{2l}^2 - 1)T_{s_i}$$

for  $1 \leq i \leq n-1$ . Consequently, we have

$$\mathbf{H}_{k,\xi} \cong \mathbf{H}_{2,n}^{(\mathbf{u})} \quad \text{where} \quad \mathbf{u} = (u_1, u_2), \quad u_2 - u_1 \equiv d \pmod{l}.$$

**Theorem 8.5** (The equal parameter case; Jacon [55, Theorem 3.2.3]). *In the above setting, assume that  $a = b = 1$ ,  $f_n(1, 1) = 0$  and  $\xi \neq 1$ . Then  $\xi$  has even order  $l \geq 2$  and*

$$\mathcal{B}_{k,\xi} := \{E^\lambda \mid \lambda \in \Lambda_{2,n}^{(\mathbf{u})}\}$$

*is a canonical basic set for  $\mathbf{H}_{k,\xi}$ , where  $\mathbf{u} = (1, l/2)$ . See (7.8) for an explicit description of  $\Lambda_{2,n}^{(\mathbf{u})}$ .*

(Note that, in this case, we have  $d + 1 \equiv l/2 \pmod{l}$ .)

**Theorem 8.6** (The case  $b = 0$ ; Jacon [55, Theorem 3.2.5]). *In the above setting, assume that  $b = 0$ ,  $a = 1$ ,  $f_n(1, 0) = 0$  and  $\xi \neq 1$ . Then  $\xi$  has even order  $l \geq 2$  and*

$$\mathcal{B}_{k,\xi} := \{E^\lambda \mid \lambda \in \Lambda_{2,n}^{(\mathbf{u})}\}$$

*is a canonical basic set for  $\mathbf{H}_{k,\xi}$ , where  $\mathbf{u} = (0, l/2)$ . See (7.8) for an explicit description of  $\Lambda_{2,n}^{(\mathbf{u})}$ .*

(Note that, in this case, we have  $d \equiv l/2 \pmod{l}$ .)

The above result yields a description of the canonical basic set for Iwahori–Hecke algebras of type  $D_n$  ( $n \geq 2$ ). Indeed, let  $W'_n$  be the subgroup of  $W_n$  generated by  $s_0, s_1, \dots, s_{n-1}$  where  $s_0 = ts_1t$ . As already pointed out in

Example 4.9,  $W'_n$  is a Coxeter group of type  $D_n$ . Furthermore, let  $L'$  be the restriction of the weight function  $L$  in Theorem 8.6 to  $W'_n$ ; then  $L'(s_i) = 1$  for  $0 \leq i \leq n-1$ . We denote the corresponding Iwahori–Hecke algebra by  $\mathbf{H}'$ . Let  $\mathcal{B}'_{k,\xi} \subseteq \text{Irr}_{\mathbb{Q}}(W'_n)$  be the canonical basic set for  $\mathbf{H}'_{k,\xi}$ . By [32, Theorem 5.5], we have

$$\mathcal{B}'_{k,\xi} = \{E' \in \text{Irr}_{\mathbb{Q}}(W'_n) \mid E' \text{ occurs in the restriction of some } E \in \mathcal{B}_{k,\xi}\},$$

where  $\mathcal{B}_{k,\xi}$  is as in Theorem 8.6. Using the information on the restriction of modules from  $W_n$  to  $W'_n$  in Example 4.9, this yields:

**Theorem 8.7** (Jacon [55, Theorem 3.2.7] and [56]). *Let  $W'_n$  be a Coxeter group of type  $D_n$  ( $n \geq 2$ ) and  $L'(s_i) = 1$  for  $0 \leq i \leq n-1$ . Assume that  $\xi$  has even order,  $l \geq 2$  say. Then*

$$\begin{aligned} \mathcal{B}'_{k,\xi} = & \{E^{[\lambda,\mu]} \mid (\lambda, \mu) \in \Lambda_{2,n}^{(\mathbf{u})}, \lambda \neq \mu\} \\ & \cup \{E^{[\lambda,\pm]} \mid n \text{ even and } \lambda \vdash n/2 \text{ is } l/2\text{-regular}\} \end{aligned}$$

is a canonical basic set for  $\mathbf{H}'_{k,\xi}$ , where  $\mathbf{u} = (0, l/2)$ .

In Table 3, we consider the case where  $n = 3$ ,  $\xi = -1$ ,  $a = 1$  and  $b \geq 0$  is even. Note that, for any such  $b$ , we have  $\xi^b = 1$  and so we are always dealing with the same algebra  $\mathbf{H}_{k,\xi}$ . In the first matrix, the rows are just ordered according to increasing value of  $\dim E^\lambda$ . Now, depending on which value of  $b$  we take, we obtain a different canonical basic set (indicated by “ $\rightarrow$ ”). The one for  $b = 0$  is given by Theorem 8.6, and it yields a canonical basic set for type  $D_3$ ; the one for  $b = 4$  corresponds to the “asymptotic case” in Example 6.9, and it yields the indexing set  $\Lambda_{2,3}^\circ$  (arising from the Specht module theory). It turns out that this is also obtained for  $b = 2$ .

In the above two results, we only considered the cases where  $a = 1$  and  $b \in \{0, 1\}$ . But similar arguments apply to all choices of  $a, b \geq 0$  where  $f_n(a, b) = 0$ , and this yields explicit combinatorial descriptions of canonical basic sets in terms of the sets  $\Lambda_{r,n}^{(\mathbf{u})}$  arising from crystal graphs, for suitable values of  $\mathbf{u}$ . For the details, see [45].

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TABLE 3. Decomposition numbers for  $B_3$  with  $\xi^b = 1$ ,  $\xi = -1$ .

$\lambda$	$\dim E^\lambda$	$[E : M]$
$(3, \emptyset)$	1	1 . . .
$(\emptyset, 3)$	1	. 1 . .
$(111, \emptyset)$	1	1 . . .
$(\emptyset, 111)$	1	. 1 . .
$(21, \emptyset)$	2	. . 1 .
$(\emptyset, 21)$	2	. . . 1
$(1, 2)$	3	1 . . 1
$(2, 1)$	3	. 1 1 .
$(11, 1)$	3	. 1 1 .
$(1, 11)$	3	1 . . 1

$b = 0$	$\alpha_\lambda$	$[E : M]$
$\rightarrow (3, \emptyset)$	0	1 . . .
$\rightarrow (\emptyset, 3)$	0	. 1 . .
$\rightarrow (1, 2)$	1	1 . 1 .
$\rightarrow (2, 1)$	1	. 1 . 1
$(21, \emptyset)$	2	. . . 1
$(\emptyset, 21)$	2	. . 1 .
$(11, 1)$	3	. 1 . 1
$(1, 11)$	3	1 . 1 .
$(111, \emptyset)$	6	1 . . .
$(\emptyset, 111)$	6	. 1 . .

$b = 2$	$\alpha_\lambda$	$[E : M]$
$\rightarrow (3, \emptyset)$	0	1 . . .
$\rightarrow (21, \emptyset)$	1	. 1 . .
$\rightarrow (2, 1)$	2	. 1 1 .
$(11, 1)$	3	. 1 1 .
$(111, \emptyset)$	3	1 . . .
$(\emptyset, 3)$	3	. . 1 .
$\rightarrow (1, 2)$	3	1 . . 1
$(1, 11)$	6	1 . . 1
$(\emptyset, 21)$	7	. . . 1
$(\emptyset, 111)$	12	. . 1 .

$b = 4$	$\alpha_\lambda$	$[E : M]$
$\rightarrow (3, \emptyset)$	0	1 . . .
$\rightarrow (21, \emptyset)$	1	. 1 . .
$(111, \emptyset)$	3	1 . . .
$\rightarrow (2, 1)$	4	. 1 1 .
$(11, 1)$	5	. 1 1 .
$\rightarrow (1, 2)$	7	1 . . 1
$(\emptyset, 3)$	9	. . 1 .
$(1, 11)$	10	1 . . 1
$(\emptyset, 21)$	13	. . . 1
$(\emptyset, 111)$	18	. . 1 .

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